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THE METHOD OF SPHERICAL HARMONICS FOR INTEGRAL TRANSFORMS ON A SPHERE

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Abstract. The integral convolution-type equations of the first kind on the sphere are important for the geometric tomography. They have been studied by many researchers. In this paper, we consider the uniqueness and stability of solutions of such equations. We prove the uniqueness of the solution for the equation with the kernel of convolution type and obtain a formula for the average value of a function on a subsphere. The latter is used for the deriving of the inversion formula of Radon spherical transformation on sphere. For the Blaschke-Levy equation and for the convolution type singular integral equations of the linear transfer theory, the uniqueness theorems are proved and find estimates for the stability of solutions are found. In all the cases we use the expansion of a function into series of spherical harmonics.

Keywords: geometric tomography, convolution-type equations of the first kind, uniqueness, stability, Radon spherical transform, inversion formula, Blaschke-Levy equation, equation with a singular kernel.

1. Introduction and preliminaries

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $|S^{n-1}|$ be its volume, $u, v \in S^{n-1}$, $u^\perp$ be the orthogonal complementary subspace for u, $S_u^{n-2} = S^{n-1} \cap u^\perp$ be the big subsphere, $\langle , \rangle$ be the inner product and σ_{n-1} be the spherical $(n-1)$-dimensional Lebesgue measure on S^{n-1}. In the geometric tomography, equations of the first kind with the following convolution type integral operators on a sphere are important:

$$(Kz)(u) = \int_{S^{n-1}} K(\langle u, v\rangle) z(v) \sigma_{n-1}(dv), u \in S^{n-1}.$$

In the cases $K(\langle u, v\rangle) = \delta(\langle u, v\rangle)$, where $\delta(t)$ is the delta-function at zero, $K(\langle u, v\rangle) = |\langle u, v\rangle|$, $K(\langle u, v\rangle) = |\langle u, v\rangle|^\alpha$, $K(\langle u, v\rangle) = \chi(\langle u, v\rangle)$, where $\chi(t)$ is Heaviside's function, the integral transforms are well-known. They are called Radon spherical transform $\mathcal{R}z$, cosine-transform $\mathcal{C}z$, α-cosine-transform $\mathcal{C}_\alpha z$, and hemispherical transform $\mathcal{H}z$, respectively.

The transforms above have the following geometrical meaning: if $(n-1)z(v) = \rho^{n-1}(v)$, where $\rho(v)$, is the radial function of a star body $\mathbf{B}$, then the Radon transform $(\mathcal{R}z)(u)$ is the volume of the intersection of $\mathbf{B}$ with the subspace $u^\perp$. If

$z(v)$ is the product of principal curvature radii of the closed smooth convex surface $\partial\mathbf{B}$ in a point with normal v, then $2(\mathcal{C}z)(u)$ is the $(n-1)$-dimensional volume of the projection of the convex body $\mathbf{B}$ onto the subspace $u^{\perp}$, and $(\mathcal{H}z)(u)$ is the area of the "lit" part of the surface $\partial\mathbf{B}$. Thus the inversion formulas for the integral transforms and the uniqueness and stability of the solution of the integral equations are of great interest.

Integral transforms with the kernels $|\langle u,v\rangle|, |\langle u,v\rangle|^{\alpha}, \chi(\langle u,v\rangle)$ are closely related to the Radon spherical transform. For example, the cosine transformation $\mathcal{C}$ is connected with the Radon transform $\mathcal{R}$ via the Laplace-Beltrami operator Δ_S on S^{n-1} due to the identity $\Box\mathcal{C} = \mathcal{R}$, where $\Box = (\Delta_S + n - 1)/(2 \cdot |S^{n-2}|)$ [10, 11]. The expansion of functions into series of spherical harmonics is a powerful tool in the investigation of the convolution-type equations (see, for example, [18]). Thus the following theorem is useful.

Theorem 1 (Funk-Hecke theorem, [3]). *Let $K(t)$ be a bounded measurable function on $[-1,1]$ and $Y_k(u)$ be a spherical harmonic of order k. Then*

$$\int_{S^{n-1}} K(\langle u,v\rangle) Y_k(v)\sigma_{n-1}(dv) = \lambda_k Y_k(u), \quad u \in S^{n-1}, \tag{1}$$

where

$$\lambda_k = \frac{|S^{n-2}|\Gamma(n-2)\Gamma(k+1)}{\Gamma(k+n-2)} \int_{-1}^{1} K(t)\, C_k^{(n/2-1)}(1-t^2)^{\frac{n-3}{2}} dt \tag{2}$$

and $C_k^{(n/2-1)}(t)$ are the Gegenbauer polynomials.

The Funk-Hecke formula can be extended onto the case of the kernel $K(t) \in L_1[-1,1]$ (see [3]).

In this paper, we apply the method of spherical harmonics to the problem of uniqueness for the solutions of convolution-type equations of the first kind. We got the inversion formula for the spherical Radon transform on S^3 and proved estimates of the solutions stability of α-cosine transform and equation with singular kernel.

2. Uniqueness of the solution of the equation for measure

We consider the equation of the first kind

$$f(u) = \int_{S^{n-1}} K(\langle u,v\rangle)\mu(dv), \tag{3}$$

where μ is the unknown signed measure on S^{n-1}. Let $\{Y_k(u)\}$ be a complete system of spherical functions on S^{n-1} and $\mathcal{M}$ be Banach space of signed measures (charges) on S^{n-1}. Let us introduce moments of the signed measure with respect to the system $\{Y_k(u)\}$:

$$\mu_k = \int_{S^{n-1}} Y_k(u)\mu(du). \tag{4}$$

Lemma 1. *The signed measure μ on S^{n-1} is uniquely defined by its moments μ_k, $k = 0, 1, 2, \ldots,$ with respect to system $\{Y_k(u)\}$.*

Proof. Let μ_1 and μ_2 be two measures whose moments coincide. Set $\mu = \mu_1 - \mu_2$. It is sufficient to prove that the equalities

$$\int_{S^{n-1}} Y_k(u)\mu(du) = 0,\ k = 0, 1, 2, \dots,$$

imply $\mu = 0$. Any polynomial is a linear combination of spherical harmonics on S^{n-1}. It follows from Weierstrass's Theorem that the linear span of spherical harmonics is dense in the Banach space of all continuous functions on the sphere

$$\int_{S^{n-1}} \varphi(u)\mu(du) = 0$$

for any continuous function $\varphi(u) \in C(S^{n-1})$. Thus $\mu = 0$, since the space of measures is dual to the space of continuous functions. ■

Theorem 2. *If* $K(\langle u, v\rangle) \in L_1[-1, 1]$ *and the system of its eigenfunctions is complete in* $L_1(S^{n-1})$*, then the equation* (3) *admits at most one solution in* $\mathcal{M}$*.*

Proof. Multiplying equation (3) by $Y_k(u)$, integrating it over the Lebesgue measure σ_{n-1}, and applying Fubini's theorem, we get:

$$\int_{S^{n-1}} f(u)Y_k(u)\sigma_{n-1}(du) = \int_{S^{n-1}} Y_k(u) \left(\int_{S^{n-1}} K(\langle u, v\rangle)\mu(dv) \right) \sigma_{n-1}(du) =$$
$$= \int_{S^{n-1}} \sigma_{n-1}(du) \int_{S^{n-1}} K(\langle u, v\rangle)Y_k(u)\mu(dv) =$$
$$= \int_{S^{n-1}} \left(\int_{S^{n-1}} K(\langle u, v\rangle)Y_k(u)\sigma_{n-1}(du) \right) \mu(dv).$$

According to Funk-Hecke formula (1),

$$\int_{S^{n-1}} K(\langle u, v\rangle)Y_k(v)\sigma_{n-1}(dv) = \lambda_k Y_k(u), \quad k = 0, 1, 2, \dots,$$

where k same for any spherical harmonics of order k, k = 0,1,2,. . . . This equality together with (4) imply

$$f_k = \int_{S^{n-1}} f(u)Y_k(u)\sigma_{n-1}(du) = \lambda_k \mu_k,\ k = 0, 1, 2, \dots\ .$$

Since the system of eigenfunctions is complete, $\lambda_k \neq 0$. Hence the Fourier coefficients f_k of the function $f(u)$ uniquely define the moments of μ by the equality $\mu_k = f_k/\lambda_k$, $k = 0, 1, 2, \dots$. Applying Lemma 1, we conclude that the measure μ is uniquely determined by its moments. ■

3. The average of a function over subspheres and spherical transform of harmonics

Let u be the pole of S^{n-1}, $v \in S^{n-1}$ and $\gamma = \widehat{(u, v)}$. Point $v \in S^{n-1}$ can be presented as $v = (v' \sin\gamma, \cos\gamma)$, where $v' \in S_u^{n-2} = S^{n-1} \cap u^{\perp}$.

The Lebesgue measure σ_{n-1} on sphere S^{n-1} and Lebesgue measure $\sigma_{n-2,\gamma}$ on subsphere $S_\gamma^{n-2} = \{v \in S^{n-1} : \langle u, v\rangle = \cos\gamma\}$ are subject to the equality $\sigma_{n-1}(dv) = \sin^{n-2}\gamma\, d\gamma\, \sigma_{n-2,\gamma}(dv'), 0 \leqslant \gamma \leqslant \pi/2$ [12]. For a function $f(v)$ on S^{n-1} we define its average value on the subsphere S_γ^{n-2} as

$$\widetilde{f}(\gamma, u) = \frac{1}{|S_\gamma^{n-2}|}\int_{S_\gamma^{n-2}} f(v)\,\sigma_{n-2,\gamma}(dv).$$

Since $|S_\gamma^{n-2}| = \sin^{n-2}\gamma|S^{n-2}|$ and $\sigma_{n-2,\gamma}(dv)$=$\sigma_{n-2,\gamma}(\sin\gamma dv')$=$\sin^{n-2}\gamma\,\sigma_{n-2}(dv')$, if $\gamma = const$, $v' \in S_u^{n-2}$, we have

$$\begin{aligned}\widetilde{f}(\gamma, u) &= \frac{1}{|S_\gamma^{n-2}|}\int_{S_\gamma^{n-2}} f(v)\,\sigma_{n-2,\gamma}(dv) = \\ &= \frac{1}{\sin^{n-2}\gamma|S^{n-2}|}\int_{S_u^{n-2}} f(v'\sin\gamma, \cos\gamma)\sin^{n-2}\gamma\,\sigma_{n-2}(dv') = \\ &= \frac{1}{|S^{n-2}|}\int_{S_u^{n-2}} f(v'\sin\gamma, \cos\gamma)\,\sigma_{n-2}(dv').\end{aligned} \tag{5}$$

We use the δ-function to obtain the formula for the average. Let us consider the integral

$$\int_{\mathbb{R}^n} \delta(|y|-1)\cdot\delta(\langle x/|x|, y\rangle - \cos\gamma)f(y)dy,$$

where $\delta(|y|-1)\cdot\delta(\langle x/|x|, y\rangle - \cos\gamma)$ is the direct product of the delta functions and $f(y)$ is the smooth finite continued of $f(v), v \in S^{n-1}$, in the area $y \in \mathbb{R}^n : 0 < r < |y| < R$, $0 < r < 1 < R$.

According to the formula [5],

$$\int_{\mathbb{R}^n} \delta(P(x))f(x)dx = \int_{P(x)=0} \frac{f(x)d\sigma_x}{|DP(x)|},$$

where $P(x) = \{p_1(x), p_2(x), \ldots, p_k(x)\} = \{0, 0, \ldots, 0\}$ is the $(n-k)$-surface in $\mathbb{R}^n$, $|DP| = \sqrt{det\langle\nabla p_i, \nabla p_j\rangle}$ is the square root of the Gram determinant of the vectors $\{\nabla p_1(x), \nabla p_2(x), \ldots, \nabla p_k(x)\}$, $d\sigma$ is the element of surface area $P(x) = \overline{0}$, for $p_1(y) = \delta(|y|-1)$, $p_2(y) = \delta(\langle x/|x|, y\rangle - \cos\gamma)$ we get:

$$\int_{\mathbb{R}^n} \delta(|y|-1)\cdot\delta(\langle x/|x|, y\rangle - \cos\gamma)f(y)dy = \int_{|y|=1, \langle x/|x|, y\rangle=\cos\gamma} \frac{f(y)\sigma_{n-2,\gamma}(dy)}{\sqrt{1-\frac{\langle x,y\rangle^2}{|x|^2|y|^2}}} =$$

$$= \int_{|y|=1, \langle x/|x|, y\rangle=\cos\gamma} \frac{f(y)\sigma_{n-2,\gamma}(dy)}{\sin\gamma}.$$

Therefore,

$$
\begin{aligned}
\widetilde{f}(\gamma,u) &= \frac{1}{\sin^{n-2}\gamma|S^{n-2}|}\int_{\langle u,v\rangle=\cos\gamma} f(v)\,\sigma_{n-2,\gamma}(dv) = \\
&= \frac{1}{\sin^{n-3}\gamma|S^{n-2}|}\int_{\langle u,v\rangle=\cos\gamma} \frac{f(v)\,\sigma_{n-2,\gamma}(dv)}{\sin\gamma} = \\
&= \frac{1}{\sin^{n-3}\gamma|S^{n-2}|}\int_{\mathbb{R}^n} \delta(|y|-1)\cdot\delta(\langle x/|x|,y\rangle-\cos\gamma)f(y)dy\Big|_{x=u} = \qquad (6)\\
&= \frac{1}{\sin^{n-3}\gamma|S^{n-2}|}\int_{S^{n-1}} \delta(\langle x/|x|,y\rangle-\cos\gamma)f(y)\sigma_{n-1}(dy)\Big|_{x=u}.
\end{aligned}
$$

To find out the average value of spherical harmonics $Y_k(u)$, let us use the Funk-Hecke formula and (6). We have:

$$
\begin{aligned}
\widetilde{Y_k}(\gamma,u) &= \frac{1}{\sin^{n-3}\gamma|S^{n-2}|}\int_{S^{n-1}} \delta(\langle x/|x|,v\rangle-\cos\gamma)Y_k(v)\sigma_{n-1}(dv)\Big|_{x=u} = \\
&= \frac{Y_k(u)}{\sin^{n-3}\gamma|S^{n-2}|}\cdot\frac{|S^{n-2}|\Gamma(n-2)\Gamma(k+1)}{\Gamma(k+n-2)}\cdot\int_{-1}^{1}\delta(t-\cos\gamma)\,C_k^{(n/2-1)}(t)(1-t^2)^{\frac{n-3}{2}}dt = \\
&= \frac{\Gamma(n-2)\Gamma(k+1)}{\sin^{n-3}\gamma\Gamma(k+n-2)}\cdot C_k^{(n/2-1)}(\cos\gamma)\sin^{n-3}\gamma Y_k(u) = \\
&= \frac{\Gamma(n-2)\Gamma(k+1)}{\Gamma(k+n-2)}\cdot C_k^{(n/2-1)}(\cos\gamma)Y_k(u).
\end{aligned}
$$

Thus, the average value of the spherical harmonic $Y_k(u)$ over the subsphere $S_\gamma^{n-2}=\{v\in S^{n-1}:\langle u,v\rangle=\cos\gamma\}$ is equal to

$$
\widetilde{Y_k}(\gamma,u) = \frac{\Gamma(n-2)\Gamma(k+1)}{\Gamma(k+n-2)}C_k^{(n/2-1)}(\cos\gamma)Y_k(u). \qquad (7)
$$

For the spherical Radon transform of the spherical harmonics $Y_{2k}(u)$, formula (7) means that

$$
(\mathcal{R}Y_{2k})(u) = \frac{1}{|S^{n-2}|}\int_{S_u^{n-2}} Y_{2k}(v)\sigma_{n-2}(dv) = \frac{\Gamma(n-2)\Gamma(2k+1)}{\Gamma(2k+n-2)}C_{2k}^{(n/2-1)}(0)Y_{2k}(u). \qquad (8)
$$

4. The inversion formula for Radon spherical transform on S^3

The results concerning the Radon transform on $\mathbb{R}^n$ at Riemannian manifolds of negative and positive curvature are given in the monographs [6], [9], [14] and in the article of [17] B. Rubin. The central point of the Radon transform is the reconstruction of a function by its integrals on submanifolds. Starting with the classical results by Funk [13], Radon [16], a lot of different variants of inversion formulas for Radon spherical transform and the related integral transforms were obtained. The history of the inversion formulas can be found in the article [17] by B. Rubin. In the case $n \geqslant 3$, the inversion formula for the spherical Radon transform $\mathcal{R}$ was obtained by Helgason in 1959, but it was not published until 1990. It is presented in [14], Theorem 3.13, p. 54. Note that the Radon transform is used in the solution of the inverse problems of the scattering theory [15]. The problem is to restore the shape, size, and the electromagnetic parameters of the scattering body. Our goal is to derive the inversion formula for the spherical Radon on S^3 by the method of spherical harmonics.

Theorem 3. *If $z(u)$ is a sufficiently smooth even function on S^3, then the inversion formula for the spherical Radon transform on S^3*

$$(\mathcal{R}z)(u) = \frac{1}{4\pi}\int_{S^3\cap u^\perp} z(v)\,\sigma_2(dv)$$

has the following form:

$$z(u) = \frac{1}{2\pi}\frac{d^2}{dt^2}\int_{\langle u,v\rangle^2>t}(\mathcal{R}z)(v)|\langle u,v\rangle|\,\sigma_3(dv)\bigg|_{t=0}.$$

Proof. The average of the spherical harmonic $Y_{2k}(u)$ on the subsphere $S_u^2 = S^3\cap u^\perp$ u (i.e., the spherical Radon transform on S^3) is given by the formula (8). If $n=4, C_{2k}^{(1)}(0) = (-1)^k$, hence

$$(\mathcal{R}Y_{2k})(u) = \widetilde{Y}_{2k}(\pi/2,u) = \frac{\Gamma(2)\Gamma(2k+1)}{\Gamma(2k+2)}C_{2k}^{(1)}(0)Y_{2k}(u) = \frac{(-1)^k}{2k+1}Y_{2k}(u). \tag{9}$$

We claim that

$$\frac{1}{2\pi}\frac{d^2}{dt^2}\int_{\langle u,v\rangle^2>t}(\mathcal{R}Y_{2k})(v)|\langle u,v\rangle|\,\sigma_3(dv)\bigg|_{t=0}$$

coincides with $Y_{2k}(u)$.

We transform the integral and use the formulas (5) and (7):

$$\frac{1}{2\pi}\frac{d^2}{dt^2}\int_{\langle u,v\rangle^2>t}(\mathcal{R}Y_{2k})(v)|\langle u,v\rangle|\,\sigma_3(dv)\bigg|_{t=0} =$$
$$= \frac{2(-1)^k}{2\pi(2k+1)}\frac{d^2}{dt^2}\int_{\langle u,v\rangle>\sqrt{t}}Y_{2k}(v)|\langle u,v\rangle|\,\sigma_3(dv)\bigg|_{t=0} =$$
$$= \frac{(-1)^k}{\pi(2k+1)}\frac{d^2}{dt^2}\int_{\langle u,v\rangle>\sqrt{t}}Y_{2k}(v'\sin\gamma,\cos\gamma)\sin^2\gamma\,\cos\gamma\,\sigma_2(dv')d\gamma\bigg|_{t=0} = \frac{4\pi(-1)^k}{\pi(2k+1)}\cdot$$
$$\cdot\frac{d^2}{dt^2}\int_0^{\arccos\sqrt{t}}\sin^2\gamma\,\cos\gamma\left(\frac{1}{4\pi}\int_{S_u^{n-2}}Y_{2k}(v'\sin\gamma,\cos\gamma)\,\sigma_2(dv')\right)d\gamma\bigg|_{t=0} = \tag{10}$$
$$= \frac{4(-1)^k}{2k+1}\,\frac{d^2}{dt^2}\int_0^{\arccos\sqrt{t}}\widetilde{Y}_{2k}(\gamma,u)\sin^2\gamma\,\cos\gamma\,d\gamma\bigg|_{t=0} =$$
$$= \frac{4(-1)^k}{(2k+1)^2}\,\frac{d^2}{dt^2}\int_0^{\arccos\sqrt{t}}C_{2k}^{(1)}(\cos\gamma)\sin^2\gamma\,\cos\gamma\,d\gamma\bigg|_{t=0}\cdot Y_{2k}(u).$$

Let $I(t)$ denote the second derivative of the integral. We find it by differentiating with the respect to the parameter t.

$$I(t) = \frac{d^2}{dt^2}\int_0^{\arccos\sqrt{t}} C_{2k}^{(1)}(\cos\gamma)\sin^2\gamma\,\cos\gamma\,d\gamma = \frac{d}{dt}\left[-\frac{C_{2k}^{(1)}(\sqrt{t})(1-t)\sqrt{t}}{2\sqrt{t}\sqrt{1-t}}\right]' =$$
$$= -\frac{1}{2}\left[\sqrt{1-t}\,C_{2k}^{(1)}(\sqrt{t})\right]' = \frac{C_{2k}^{(1)}(\sqrt{t})}{4\sqrt{1-t}} - \frac{\sqrt{1-t}}{2}\cdot\frac{dC_{2k}^{(1)}(\sqrt{t})}{dt} =$$
$$= \frac{C_{2k}^{(1)}(\sqrt{t})}{4\sqrt{1-t}} - \frac{\sqrt{1-t}}{4\sqrt{t}}\cdot\frac{dC_{2k}^{(1)}(\sqrt{t})}{d\sqrt{t}}.$$

For the Gegenbauer polynomials there is the following differentiation formula [3]:

$$(1-t^2)\frac{dC_{2k}^{(1)}(t)}{dt} = (2k+2)tC_{2k}^{(1)}(t) - (2k+1)C_{2k+1}^{(1)}(t).$$

Applying it, we get

$$I(t) = \frac{C_{2k}^{(1)}(\sqrt{t})}{4\sqrt{1-t}} - \frac{\sqrt{1-t}}{4\sqrt{t}(1-t)}\left[(2k+2)\sqrt{t}C_{2k}^{(1)}(\sqrt{t}) - (2k+1)C_{2k+1}^{(1)}(\sqrt{t})\right] =$$
$$= \frac{C_{2k}^{(1)}(\sqrt{t})}{4\sqrt{1-t}} - \frac{(2k+2)C_{2k}^{(1)}(\sqrt{t})}{4\sqrt{1-t}} + \frac{(2k+1)C_{2k+1}^{(1)}(\sqrt{t})}{4\sqrt{t}\sqrt{1-t}} =$$
$$= -\frac{(2k+1)C_{2k}^{(1)}(\sqrt{t})}{4\sqrt{1-t}} + \frac{(2k+1)C_{2k+1}^{(1)}(\sqrt{t})}{4\sqrt{t}\sqrt{1-t}}.$$

It follows from the equality [3]

$$C_{2k+1}^{(1)}(s) = \sum_{m=0}^{k}\frac{(-1)^m(1)_{2k+1-m}(2s)^{2k+1-2m}}{m!(2k+1-2m)!} = \frac{(1)_{2k+1}(2s)^{2k+1}}{(2k+1)!} + \ldots +$$
$$+ \frac{(-1)^{k-1}(1)_{k+2}(2s)^3}{3!(k-1)!} + \frac{(-1)^k(1)_{k+1}(2s)^1}{1!(k)!},$$

where $(1)_p = \Gamma(p+1)/\Gamma(1) = p!$ that

$$\left.\frac{C_{2k+1}^{(1)}(\sqrt{t})}{\sqrt{t}}\right|_{t=0} = \frac{2(-1)^k(1)_{k+1}}{1!k!} = \frac{2(-1)^k\Gamma(k+2)}{\Gamma(1)k!} = \frac{2(-1)^k(k+1)!}{k!} =$$
$$= 2(-1)^k(k+1).$$

Thus, we get $I(0)$:

$$I(0) = \frac{(2k+1)\left[-C_{2k}^{(1)}(0) + 2(-1)^k(k+1)\right]}{4} = \frac{(-1)^k(2k+1)^2}{4} \tag{11}$$

because $C_{2k}^{(1)}(0) = (-1)^k$.

According to formula (9), (10), (11)

$$\frac{1}{2\pi}\frac{d^2}{dt^2}\int_{\langle u,v\rangle^2>t}(\mathcal{R}Y_{2k})(v)|\langle u,v\rangle|\,\sigma_3(dv)\bigg|_{t=0} = \frac{4(-1)^k}{(2k+1)^2}\cdot\frac{(-1)^k(2k+1)^2}{4}Y_{2k}(u) =$$
$$= Y_{2k}(u),\ k=0,1,2,\dots\ .$$

This proves the inversion formula for the spherical harmonics $Y_{2k}(u),\ k=0,1,2,\dots$.

Let $z(u)$ be a sufficiently smooth even function on S^3 . Let us expand it into series of spherical harmonics of even order:

$$z(u)=\sum_{k=0}^{\infty}Y_{2k}(u).$$

Its Radon transform is

$$(\mathcal{R}z)(u)=\sum_{k=0}^{\infty}(\mathcal{R}Y_{2k})(u).$$

Hence

$$\frac{1}{2\pi}\frac{d^2}{dt^2}\int_{\langle u,v\rangle^2>t}(\mathcal{R}z)(v)|\langle u,v\rangle|\,\sigma_3(dv)\bigg|_{t=0} =$$
$$=\sum_{k=0}^{\infty}\frac{1}{2\pi}\frac{d^2}{dt^2}\int_{\langle u,v\rangle^2>t}(\mathcal{R}Y_{2k})(v)|\langle u,v\rangle|\,\sigma_3(dv)\bigg|_{t=0} = \sum_{k=0}^{\infty}Y_{2k}(u)=z(u)$$

and we get the inversion formula for the spherical Radon transform of an even smooth function on S^3. ■

5. The estimate of the stability of the equation Blaschke-Levy solution

Let us consider the Blaschke-Levy equation (α-cosine transform)

$$f(u)=\frac{1}{4\pi}\int_{S^2}|\langle u,v\rangle|^{\alpha}z(v)\sigma_2(dv),\ u\in S^2, \tag{12}$$

where $\sigma_2(\cdot)$ is the Lebesgue measure on S^2, $z(u)$ is an even function on S^2, $\alpha>-1$, $\alpha\neq 0,2,4,\dots,2m,\dots,m\in\mathbb{N}$. We shall formulate the restrictions on $f(u)$ later assuming now that $f(u)$ is a sufficiently smooth even function.

Eigenvalues of the operator (12) can be found by the Funk-Hecke formula (2):

$$\lambda_k=\frac{1}{2}\int_{-1}^{1}|t|^{\alpha}C_k^{(1/2)}(t)dt.$$

The Gegenbauer polynomials $C_k^{(1/2)}(t)$ coincide with the Legendre polynomials $P_k(t)$, which satisfy the condition $P_k(-t)=(-1)^kP_k(t)$. Consequently, eigenfunctions of the equation (12) are spherical harmonics $Y_{2k}(u)$ of even order and the corresponding eigenvalues λ_{2k} are determined by the integral

$$\lambda_{2k} = \int_0^1 t^\alpha P_{2k}(t)dt.$$

This integral converges at $\alpha > -1$ and its value is given in [4]:

$$\lambda_{2k} = \frac{\alpha(\alpha-2)(\alpha-4)\cdot\ldots\cdot(\alpha-2k+2)(\alpha-2k+4)}{(\alpha+1)(\alpha+3)\cdot\ldots\cdot(\alpha+2k-1)(\alpha+2k+1)}.$$

Expanding the function f(u) over the system of spherical functions $\{Y_{2k}(u)\}$

$$f(u) = \sum_{k=0}^{\infty} Y_{2k}(u),$$

where, by the Laplace formula [20],

$$Y_{2k}(u) = \frac{4k+1}{4\pi}\int_{S^2} P_{2k}(\langle u, v\rangle) f(v)\sigma_2(dv).$$

we get the solution of the equation (5.1) in a form of series:

$$z(u) = \sum_{k=0}^{\infty} \alpha_{2k} Y_{2k}(u),$$

where

$$\alpha_{2k} = \frac{1}{\lambda_{2k}} = \frac{(\alpha+1)(\alpha+3)\cdot\ldots\cdot(\alpha+2k-1)(\alpha+2k+1)}{\alpha(\alpha-2)(\alpha-4)\cdot\ldots\cdot(\alpha-2k+2)(\alpha-2k+4)}.$$

If $\alpha > -1$ is not an even nonnegative number, then the equation (5.1) has not more than one solution.

Let us estimate the stability. Since the Legendre polynomials $P_{2k}(\langle u, v\rangle)$ are spherical functions, we have

$$P_{2k}(\langle u, v\rangle) = \frac{(-1)^\ell}{[2k(2k+1)]^\ell}\cdot \Delta_S^\ell P_{2k}(\langle u, v\rangle).$$

The surface Laplace operator S is self-adjoint. Hence

$$Y_{2k}(u) = \frac{(-1)^\ell(4k+1)}{4\pi[2k(2k+1)]^\ell}\int_{S^2} P_{2k}(\langle u, v\rangle)\Delta_S^\ell f(v)\sigma_2(dv).$$

Together with the Cauchy-Schwartz inequality and the inequality $\|P_{2k}(\langle u, v\rangle)\|_{L_2[-1,1]} \leqslant C_1 k^{-1/2}$, $C_1 = const$ [20], this implies

$$\|Y_{2k}(u)\|_{L_2(S^2)} \leqslant \frac{(4k+1)}{4\pi[2k(2k+1)]^\ell}\|\Delta_S^\ell f(u)\|_{L_2(S^2)}\cdot\|P_{2k}(\langle u, v\rangle)\|_{L_2[-1,1]} \leqslant$$
$$\leqslant C_2\cdot(2k)^{-2\ell+1/2}\|\Delta_S^\ell f(u)\|_{L_2(S^2)},\ C_2 = const.$$

It follows from the above inequality that

$$\|Y_{2k}(u)\|^2_{L_2(S^2)} \leqslant C_3 \cdot k^{1-4\ell}\, \|\Delta_S^\ell f(u)\|^2_{L_2(S^2)},\ C_3 = const.$$

Let us estimate the norm of the solution $z(u)$ of the equation (12) in $L_2(S^2)$.

$$\|z(u)\|_{L_2(S^2)} = \sqrt{\int_{S^2} \left(\sum_{k=0}^{\infty} \alpha_{2k} Y_{2k}(u)\right)^2 \sigma_2(du)} = \sqrt{\sum_{k=0}^{\infty} \alpha_{2k}^2 \|Y_{2k}(u)\|^2_{L_2(S^2)}} \leqslant$$

$$\leqslant C \sqrt{\sum_{k=0}^{\infty} \alpha_{2k}^2 k^{1-4\ell}}\ \left\|\Delta_S^\ell f(u)\right\|_{L_2(S^2)},\ C = const.$$

To analyze the series $\sum_{k=0}^{\infty} \alpha_{2k}^2 k^{1-4\ell}$ on the convergence we use the Gaussian criterion. The ratio

$$\frac{\beta_{2k}}{\beta_{2k+2}} = \frac{\alpha_{2k}^2}{\alpha_{2k+2}^2} \cdot \frac{(k+1)^{4\ell-1}}{k^{4\ell-1}} = \frac{(1-\alpha/2k)^2(1+1/k)^{4\ell-1}}{[1+(\alpha+3)/2k]^2}$$

is equivalent to

$$\frac{\beta_{2k}}{\beta_{2k+2}} = 1 + \frac{4\ell - 2\alpha - 4}{k} + \frac{\nu_k(\ell,\alpha)}{k^2},$$

where $\nu_k(\ell,\alpha)$ is bounded. By the Gaussian criterion, the series converges if $\mu = 4l - 2\alpha - 4 > 1$, i.e. if $\ell > (2\alpha+5)/4$. Thus we obtain the estimates for the solution of (12) depending on α.

Theorem 4. *If $f(u) \in C^{2\ell}(S^{n-1})$, then for any $\alpha > -1, \alpha \neq 0, 2, 4, \dots, 2m$, $m \in \mathbb{N}$, there exists the unique solution of class $L_2(S^2)$ for the equation* (12). *Moreover, if $-1 < \alpha < -0.5$, then $\|z(u)\|_{L_2(S^2)} \leqslant c_1 \|\Delta_S f(u)\|_{L_2(S^2)}$; if $(4k-1)/2 \leqslant \alpha < (4k+3)/2$, then $\|z(u)\|_{L_2(S^2)} \leqslant c_\ell \|\Delta_S^\ell f(u)\|_{L_2(S^2)}$, where $\ell = k+2$, $k = 0, 1, 2, \dots$.*

6. Equations with singular kernels on S^{n-1}

The analysis of some kinetic equations in the neutron transport theory [7] lead to the problem of the determination of the scattering indicatrix $z(v,t)$ from the equation

$$f(u,t) = \int_{S^{n-1}} K(\langle u,v\rangle) z(v,t)\, \sigma_{n-1}(dv),\ u \in S^{n-1},\ t \geqslant 0.$$

In particular, the following equation with singular kernel was obtained in [1]:

$$f(u) = \int_{S^{n-1}} \frac{z(v)\, \sigma_{n-1}(dv)}{1 - \langle u,v\rangle}.$$

We consider a more general equation

$$f(u) = \int_{S^{n-1}} \frac{z(v)\, \sigma_{n-1}(dv)}{[1 - \langle u,v\rangle]^\alpha}. \tag{13}$$

If $\alpha < (n-1)/2$, then the kernel $K(t) = (1-t)^{-\alpha}$ satisfies condition [2]

$$\int_{-1}^{1} \frac{(1-t^2)^{(n-3)/2}dt}{(1-t)^{\alpha}} = \int_{-1}^{1} (1-t)^{(n-2\alpha-3)/2}(1+t)^{(n-3)/2}dt =$$
$$= 2^{n-1-\alpha} B\left(\frac{n-1}{2}, \frac{n-2\alpha-1}{2}\right) < \infty,$$

where $B(x,y)$ is the beta-function. Thus we may use formula (2) to find the eigenvalues [19, p. 431]:

$$\lambda_k = \frac{|S^{n-2}|\Gamma(n-2)\Gamma(k+1)}{\Gamma(k+n-2)} \int_{-1}^{1} (1-t)^{(n-2\alpha-3)/2}(1+t)^{(n-3)/2} C_k^{(n/2-1)}(t)dt.$$

We use the following asymptotic formula for the Gegenbauer polynomials $C_k^{(n/2-1)}(t)$ as $k \to \infty$, which was found in the paper [8]:

$$C_k^{(n/2-1)}(t) \sim \frac{2^{n/2-1}\Gamma((n-1)/2)}{\sqrt{\pi}(n-3)!} \cdot \frac{(k+n-3)!}{k! k^{n/2-1}} (1-t^2)^{-n/4+1/2}.$$
$$\cdot \cos\left[(k+n/2-1)\arccos t + (2\pi - n\pi)/4\right], \ -1 < t < 1, \ n \geqslant 3.$$

It follows that the eigenvalues are subject to the following asymptotic formula:

$$\lambda_k \sim \frac{2^{n/2}\pi^{(n-2)/2}}{k^{\,n/2-1}} \cdot$$
$$\cdot \int_{-1}^{1} (1-t)^{n/4-\alpha-1}(1+t)^{n/4-1} \cos\left[(k+n/2-1)\arccos t + (2\pi - n\pi)/4\right] dt$$

as $k \to \infty$. The integral converges if $\alpha < n/4$.

Due to the fast oscillation of the integrand, the integral is small for large k. The stationary phase method shows that the integral is $o(1)$ (see [8]), whence $\lambda_k = o(k^{-n/2+1})$ as $k \to \infty$. Moreover, there are constant $a > 0$ and $b > 0$ such that $ak^{-n/2+1} \leqslant |\lambda_k| \leqslant bk^{-n/2+1}, k \to \infty$. Thus, the eigenfunctions of the integral operator in the right-hand side of (13) are the spherical harmonics $Y_k(u)$ and λ_k are its eigenvalues.

Let $f(u) = \sum_{k=0}^{\infty} Y_k(u)$, where (see [20, p. 489]

$$Y_k(u) = \frac{(n+2k-2)\Gamma(n/2-1)}{4\pi^{n/2}} \int_{S^{n-1}} C_k^{(n/2-1)}(\langle u,v\rangle) f(v)\sigma_{n-1}(dv)$$

by the Laplace formula, be the expansion of the function $f(u)$ into the series of spherical harmonics. Then

$$z(u) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k} Y_k(u) \tag{14}$$

is the solution to equation (13). If $f(u) \in C^{2\ell}(S^{n-1})$, then

$$Y_k(u) = \frac{(-1)^{\ell}(n+2k-2)\Gamma(n/2-1)}{4\pi^{n/2}[k(n+k-2)]^{\ell}} \int_{S^{n-1}} C_k^{(n/2-1)}(\langle u,v\rangle)\Delta_S^{\ell} f(v)\sigma_{n-1}(dv).$$

Applying the $L_2(S^{n-1})$ - estimates $\|C_k^{(n/2-1)}(\langle u,v\rangle)\|_{L_2(S^{n-1})} \leqslant c(n)k^{n/2-2}$ [20] for the Gegenbauer polynomials, the Cauchy-Schwartz inequality, and the asymptotics of eigenvalues we get:

$$\frac{|Y_k(u)|}{|\lambda_k|} \leqslant \frac{c(n)k^{n/2-1-2\ell}\|\Delta_S^\ell f\|_{L_2(S^{n-1})}}{|\lambda_k|} \leqslant c(n)k^{-2\ell+n-2}\|\Delta_S^\ell f\|_{L_2(S^{n-1})},\ \ell \geqslant 1,$$
$$c(n) = \mathrm{const}\,.$$

Consequently, $\ell \geqslant 1$ implies the uniform convergence of the series (14) and the continuity of the sum.

Thus, we obtain the following result.

Theorem 5. *If $f(u) \in C^{2\ell}(S^{n-1})$ and $\ell \geqslant \left[\frac{n-1}{2}\right] + 1$, then there exists the unique smooth solution of the equation (13). Moreover, it admits the following estimate:*

$$\|z(u)\|_{C(S^{n-1})} \leqslant c(n)\|\Delta_S^\ell f(u)\|_{L_2(S^{n-1})}.$$

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МЕТОД СФЕРИЧЕСКИХ ГАРМОНИК ДЛЯ ИНТЕГРАЛЬНЫХ ПРЕОБРАЗОВАНИЙ НА СФЕРЕ

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Аннотация. Интегральные уравнения первого рода типа свёртки на сфере S^{n-1} имеют важное значение в геометрической томографии и исследовались многими авторами. В представленной работе рассматриваются вопросы единственности и устойчивости решений таких уравнений. Доказана единственность решения относительно меры уравнения с ядром $K(\langle u, v \rangle)$ класса $L_1[-1, 1]$. Получена формула для среднего значения функции на подсферах, которая затем используется для вывода формулы обращения сферического преобразования Радона на сфере S^3. Для уравнения Бляшке-Леви и для уравнения типа свёртки с сингулярным ядром, встречающимся в линейной теории переноса, доказаны теоремы единственности и даны оценки устойчивости решений. Во всех случаях используется метод разложения функций по полной системе сферических гармоник.

Ключевые слова: геометрическая томография, уравнение первого рода типа свёртки, единственность, устойчивость, сферическое преобразование Радона, формула обращения, уравнение Бляшке-Леви, уравнение с сингулярным ядром.

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