*Mathematical Structures and Modeling 2025. N. 2 (74). PP. 38–44*

UDC 517.545, 517.962.2, 519.173
DOI 10.24147/2222-8772.2025.2.38-44

# ON THE KIRCHHOFF INDEX FOR CIRCULANT GRAPH WITH NON-FIXED JUMPS

**G.K. Sokolova**[1,2]
Ph.D. student, Assistant, e-mail: g.sokolova@g.nsu.ru

[1]Novosibirsk State University, Novosibirsk, Russia
[2]Novosibirsk State Technical University, Novosibirsk, Russia

**Abstract.** This article deals with a graph invariant called the Kirchhoff index. An explicit analytical formula for the Kirchhoff index of circulant graphs with non-fixed jumps is defined.

**Keywords:** circulant graph with non-fixed jumps, Laplacian eigenvalues, Kirchhoff index of a graph.

## Introduction

The Kirchhoff index $Kf(G)$ of a finite connected graph $G$ was originally defined by D.J. Klein and M. Randić [1] as the mean resistance distance between its vertices, in other words,

$$Kf(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij},$$

where $r_{ij}$ is the resistance distance between the vertices $v_i$ and $v_j$, i.e. $r_{ij}$ is equal to the resistance between equivalent points on an associated electrical network obtained by replacing each edge of $G$ by a unit resistor. Later the simple formula

$$Kf(G) = n\sum_{j=2}^{n}\frac{1}{\lambda_j},$$

relating the Kirchhoff index to the spectrum of the Laplace matrix was independently found in [2] by I. Gutman, B. Mohar and [3] by H.Y. Zhu, D.J. Klein and I. Lukovits. Kirchhoff indices for various graph families have been studied, for example, in [4–8]. In particular, the analytical formula for the Kirchhoff index for a circulant graph with fixed jumps was found in paper [8], where the asymptotics of this formula was also investigated.

The main goal of this paper is to find explicit analytical formulas for the Kirchhoff indices of circulant graphs with non-fixed jumps. We will present these formulas as sums of finitely many terms whose number is independent of n, and each of these terms amounts to a rational function evaluated at the roots of some fixed polynomial.

## 1. Preliminaries and preliminary results

Consider the finite connected graph $G_n$, that is, a graph containing one component of connectivity with finite sets of vertices $V(G)$ and edges $E(G)$. Suppose a graph $G_n$ allows multiple edges, but not loops. The following class of circulant graphs is considered throughout this paper.

**Definition 1.** A graph $G_n = C_{\beta n}(s_1, \ldots, s_k, \alpha_1 n, \ldots, \alpha_\ell n)$ is called a *circulant graph with non-fixed jumps* $1 \leqslant s_1 < \ldots < s_k < \left[\frac{\beta n}{2}\right]$ and $1 \leqslant \alpha_1 < \ldots < \alpha_\ell \leqslant \left[\frac{\beta}{2}\right]$ on $\beta n$ vertices if any $i$-th vertex is adjacent to vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k$ and $i \pm \alpha_1 n$, $i \pm \alpha_2 n, \ldots, i \pm \alpha_\ell n$ modulo $\beta n$. Here $\beta$ and $\ell$ are positive integers and $n$ is assumed to be sufficiently large.

Note that if $\alpha_\ell < \left[\frac{\beta}{2}\right]$, then the graph $G_n$ does not contain multiple edges. If $\alpha_\ell = \left[\frac{\beta}{2}\right]$, then any $i$-th vertex is connected to the vertex $i \pm \frac{\beta n}{2}$ modulo $\beta n$ by two parallel edges.

Let $A = \{a_{uv}\}_{u,v \in V(G)}$ be the *adjacency matrix* of $G$, where $a_{uv}$ is the number of edges between vertices $u$ and $v$ of $G$. Let us introduce a *valency matrix* $D = \{d_{vv}\}_{v \in V(G)}$, where $d_{vv}$ is a degree of the vertex $v \in V(G)$ which may be determined by $d_{vv} = \sum\limits_{u \in V(G)} a_{uv}$. Then the matrix $\mathcal{L} = D - A$ is called the *Laplace matrix* or *Laplacian* of the graph $G$.

Associate with each graph $G_n = C_{\beta n}(s_1, \ldots, s_k, \alpha_1 n, \ldots, \alpha_\ell n)$ the associated Laurent polynomial

$$L(z) = 2(k+\ell) - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}) - \sum_{m=1}^{\ell}(z^{\alpha_m n} + z^{-\alpha_m n}),$$

describing the structure of the Laplace matrix $\mathcal{L}$ of a given graph. Note that the numbering of the vertices of a circulant graph $G_n$ may be chosen in such a way that the adjacency matrix $A$ and the Laplace matrix $\mathcal{L}$ of $G_n$ are circulant. Recall that a matrix of order $n$ is called *circulant* if it has the form

$$\operatorname{circ}(x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ x_n & x_1 & x_2 & \ldots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \ldots & x_0 \end{pmatrix}.$$

This means the Laplacian of the graph $G_n$ may be defined as the matrix

$$\mathcal{L} = L(T) = 2(k+\ell)\mathbb{E} - \sum_{i=1}^{k}(T^{s_i} + T^{-s_i}) - \sum_{m=1}^{\ell}(T^{\alpha_m n} + T^{-\alpha_m n}),$$

where $T = \operatorname{circ}(0, 1, 0, \ldots, 0)$ is circulant matrix of order $\beta n$, which represents the cyclic shift operator $T : (x_1, x_2, \ldots, x_{\beta n-1}, x_{\beta n}) \to (x_2, x_3, \ldots, x_{\beta n}, x_1)$, and $\mathbb{E}$ is the identity matrix of the respective order. Fined the spectrum of the Laplace matrix. Suppose that $\lambda$ be an eigenvalue of $\mathcal{L}$ and $v$ be the respective eigenvector. As is known, then $\det(\mathcal{L}-\lambda\mathbb{E}) = 0$ and the following system of linear equations holds

$$\left[(2(k+\ell)-\lambda)\mathbb{E} - \sum_{i=1}^{k}(T^{s_i}+T^{-s_i}) - \sum_{m=1}^{\ell}(T^{\alpha_m n}+T^{-\alpha_m n})\right] v = 0. \tag{1}$$

Note that [9] the powers of the primitive root of unity $\zeta_{\beta n}^{j}$ are the eigenvalues of $T$, here $j = 0, 1, \dots, \beta n - 1$ and $\zeta_\ell = e^{i\frac{2\pi}{\ell}}$. This means that the matrix $T$ is similar to the diagonal matrix $\mathbb{T} = \mathrm{diag}(1, \zeta_{\beta n}, \dots, \zeta_{\beta n}^{\beta n-1})$ and the unit vectors $\mathrm{e}_{j+1} = (0, \dots, 0, \underbrace{1}_{j+1-\mathrm{th}}, 0, \dots, 0)$ of length $\beta n$ are Laplacian eigenvectors. The matrix of system (1) is written in diagonal form

$$\left[(2(k+\ell)-\lambda)\mathbb{E} - \sum_{i=1}^{k}(\mathbb{T}^{s_i}+\mathbb{T}^{-s_i}) - \sum_{m=1}^{\ell}(\mathbb{T}^{\alpha_m n}+\mathbb{T}^{-\alpha_m n})\right] \mathrm{e}_j = 0.$$

From this relation it follows the eigenvalues $\lambda_j$ of the Laplacian $\mathcal{L}$ are given by the formula

$$\lambda_j = 2(k+\ell) - \sum_{i=1}^{k}(\zeta_{\beta n}^{js_i}+\zeta_{\beta n}^{-js_i}) - \sum_{m=1}^{\ell}(\zeta_{\beta n}^{j\alpha_m n}+\zeta_{\beta n}^{-j\alpha_m n}) = L(\zeta_{\beta n}^{j}).$$

Recall the considered graph $G_n$ is assumed to be connected. This means that $\lambda_0 = 0$ and $\lambda_j > 0$ for $j = 1, 2, \dots, \beta n - 1$.

In conclusion of this section, we present Theorem and Lemma from the article [8] which are necessary to prove the main result of this paper.

**Theorem 1.** *The Kirchhoff index of the circulant graph $G = C_n(s_1, s_2, \dots, s_k)$ can be calculated as*

$$Kf_G = \frac{n}{12\sum\limits_{i=1}^{k} s_i^2}\left(n^2 - \frac{\sum\limits_{i=1}^{k} s_i^4}{\sum\limits_{i=1}^{k} s_i^2}\right) + \sum_{p=2}^{s_k}\frac{n^2\,\mathcal{U}_{n-1}(w_p)}{Q'(w_p)(1-\mathcal{T}_n(w_p))},$$

*where $w_p$ is a root of the polynomial $Q(w) = \sum\limits_{j=1}^{k}(2-2\mathcal{T}_{s_j}(w))$ distinct from $1$, where $\mathcal{T}_n(w)$ and $\mathcal{U}_n(w)$ are the Chebyshev polynomials of the first and the second kind respectively.*

**Lemma 1.** *Consider two nonconstant polynomials $P(w)$ and $R(w)$ of degrees $n$ and $m$, respectively. Denote the roots of $P(w)$ by $\alpha_1, \alpha_2, \dots, \alpha_n$ and the roots of $R(w)$ by $\beta_1, \beta_2, \dots, \beta_m$. Suppose that $R(w)$ lacks multiple roots and that $P(w)$ and $R(w)$ lack common roots. Then*

$$\sum_{j=1}^{n}\frac{1}{R(\alpha_j)} = -\sum_{j=1}^{m}\frac{1}{R'(\beta_j)}\frac{P'(\beta_j)}{P(\beta_j)}.$$

## 2. Kirchhoff index for circulant graph

In this section, an explicit analytical formula for the Kirchhoff index of a circulant graph with non-fixed jumps is given. The formula contains a sum whose terms amount to analytical expressions evaluated at the roots of a prescribed polynomial of degree $s_k$.

Consider the Laurent polynomial $L(z)$ for $G_n = C_{\beta n}(s_1, \ldots, s_k, \alpha_1 n, \ldots, \alpha_\ell n)$ and represent it as the sum $L(z) = P(z) + p(z^n)$ of polynomials

$$P(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}), \;\; p(z) = 2\ell - \sum_{m=1}^{\ell}(z^{\alpha_m} + z^{-\alpha_m}).$$

Introduce the following set of polynomials

$$P_u(z) = P(z) + p(\zeta_{\beta n}^{un}), \;\; u = 0, 1, \ldots, \beta n - 1, \tag{2}$$

Where it is easy to see that

$$p(\zeta_{\beta n}^{un}) = 4\sum_{m=1}^{\ell} \sin^2\left(\frac{u\alpha_m \pi}{\beta}\right).$$

Let $\mathcal{T}_n(w) = \cos n\theta$ be the Chebyshev polynomial of the first kind [10], here $\theta = \arccos w$. Since the equality $\mathcal{T}_n\left(\frac{z+z^{-1}}{2}\right) = \frac{z^n+z^{-n}}{2}$ satisfies for the Chebyshev polynomial $\mathcal{T}_n(w)$, the polynomial $P(z)$ may be written as

$$P(z) = Q(w) = \sum_{i=1}^{k}(2 - 2\mathcal{T}_{s_i}(w)),$$

where $w = \frac{z+z^{-1}}{2}$. Thus, for polynomials (2) the following representation is valid

$$P_u(z) = Q_u(w) = \sum_{i=1}^{k}(2 - 2\mathcal{T}_{s_i}(w)) + 4\sum_{m=1}^{\ell} \sin^2\left(\frac{u\alpha_m \pi}{\beta}\right).$$

Note that the roots of the polynomials $P_u(z)$ and $Q_u(z)$ are related by the following fact.

**Remark 1.** If the quantities $z_k, \frac{1}{z_k}$, for $k = 1, 2, \ldots s$, are the roots of the polynomial $P_u(z)$, then the numbers $w_k = \frac{z_k+z_k^{-1}}{2}$ are the roots of the polynomial $Q_u(w)$.

Suppose that the polynomials $L(z)$ and $Q(w)$ lack multiple zeros. Since eigenvalues of the Laplacian $\lambda_j = L(e^{i\frac{2\pi j}{\beta n}})$ of the graph $G_n$, then

$$\lambda_j = P_j(e^{i\frac{2\pi j}{\beta n}}) = Q\left(\cos\frac{2\pi j}{\beta n}\right) + 4\sum_{m=1}^{\ell} \sin^2\frac{j\alpha_m \pi}{\beta}.$$

Let us formulate the main Theorem of this article.

**Theorem 2.** *The Kirchhoff index of a graph $G_n = C_{\beta n}(s_1, \ldots, s_k, \alpha_1 n, \ldots, \alpha_\ell n)$ with jumps $1 \leqslant s_1 < \ldots < s_k < \left[\frac{\beta n}{2}\right]$ and $1 \leqslant \alpha_1 < \ldots < \alpha_\ell \leqslant \left[\frac{\beta}{2}\right]$ can be calculated using the formula*

$$Kf_{G_n} = \frac{\beta n}{12 \sum\limits_{i=1}^{k} s_i^2} \left( n^2 - \frac{\sum\limits_{i=1}^{k} s_i^4}{\sum\limits_{i=1}^{k} s_i^2} \right) + \sum_{\substack{\omega:\, Q(\omega)=0 \\ \omega \neq 1}} \frac{\beta^2 n^2 \mathcal{U}_{\beta n-1}(\omega)}{Q'(\omega)(1 - \mathcal{T}_{\beta n}(\omega))},$$

*where $T_{\beta n}(w)$ and $\mathcal{U}_{\beta n-1}(w)$ are the Chebyshev polynomials of the first and the second kind respectively.*

*Proof.* Since Laplacian eigenvalues of the graph $G_n$ are $\lambda_j = L(e^{i\frac{2\pi j}{\beta n}})$, then the Kirchhoff index is written as

$$Kf_{G_n} = \beta n \sum_{j=1}^{\beta n-1} \frac{1}{\lambda_j} = \beta n \sum_{j=1}^{\beta n-1} \frac{1}{Q(\cos\frac{2\pi j}{\beta n}) + 4\sum\limits_{m=1}^{\ell} \sin^2 \frac{j\alpha_m \pi}{\beta}}.$$

Let us introduce a substitution in the index of the series $j = \beta t + u$, then $0 \leqslant t \leqslant n-1$ and $0 \leqslant u \leqslant \beta - 1$. The Kirchhoff index may be rewritten by formula

$$Kf_{G_n} = \sum_{t=1}^{n-1} \frac{\beta n}{Q(\cos\frac{2\pi t}{n})} + \sum_{u=1}^{\beta-1}\sum_{t=0}^{n-1} \frac{\beta n}{Q(\cos\frac{2\pi(\beta t+u)}{\beta n}) + 4\sum\limits_{m=1}^{\ell} \sin^2 \frac{u\alpha_m \pi}{\beta}},$$

where the terms of type

$$\sum_{t=1}^{n-1} \frac{n}{Q(\cos\frac{2\pi t}{n})} = Kf_G$$

is Kirchhoff index $Kf(G)$ for a circulant graph with fixed jumps $G$. This means that it is necessary to calculate only the sum

$$\sum_{u=1}^{\beta-1}\sum_{t=0}^{n-1} \frac{\beta n}{Q(\cos\frac{2\pi(\beta t+u)}{\beta n}) + 4\sum\limits_{m=1}^{\ell} \sin^2 \frac{u\alpha_m \pi}{\beta}}.$$

Note that the numbers $\alpha_t^0 = \cos\frac{2\pi t}{n}$ are all roots of the polynomial $\mathcal{T}_n(w) - 1$, where $\beta = 1$ and $t = 1, 2, \ldots, n$. Then the numbers $\alpha_t^u = \cos\frac{2\pi(\beta t+u)}{\beta n}$ are roots for the polynomial $\frac{\mathcal{T}_{\beta n}(w)-1}{\mathcal{T}_n(w)-1}$. Introduce the following notations

$$P(w) = \mathcal{T}_{\beta n}(w) - 1, \quad R(w) = Q(w) + 4\sum_{m=1}^{\ell} \sin^2 \frac{u\alpha_m \pi}{\beta}.$$

Denote the roots of $P(z)$ by $\alpha_t^u$, here $t = 1, 2, \ldots, n$, and $u = 1, 2, \ldots, \beta - 1$, the roots of $R(w)$ by $\beta_j^u$ where $j = 1, 2, \ldots, q$ and $u = 1, 2, \ldots, \beta - 1$. Note that

$$R(\alpha_t^u) = Q(\alpha_t^u) + 4 \sum_{m=1}^{\ell} \sin^2 \frac{u\alpha_m \pi}{\beta} > 0,$$

in other words, the polynomial $R(w)$ does not vanish at the roots of polynomial $P(w)$. It follows from this that the polynomials $P(w)$ and $R(w)$ lack common roots. Observe $P'(w) = \beta n \mathcal{U}_{\beta n-1}(w)$ and $R'(w) = Q'(w)$. According to Lemma 1, the equality holds

$$\sum_{u=1}^{\beta-1} \sum_{t=0}^{n-1} \frac{1}{R(\alpha_t^u)} = \sum_{u=1}^{\beta-1} \sum_{j=1}^{q} \frac{\beta n \mathcal{U}_{\beta n-1}(\beta_j^u)}{Q'(\beta_j^u)(1 - \mathcal{T}_{\beta n}(\beta_j^u))},$$

Thus, we obtain

$$Kf_{G_n} = \beta Kf_G + \sum_{u=1}^{\beta-1} \sum_{j=1}^{q} \frac{\beta^2 n^2 \mathcal{U}_{\beta n-1}(\beta_j^u)}{Q'(\beta_j^u)(1 - \mathcal{T}_{\beta n}(\beta_j^u))}.$$

Combining formula (1) for $Kf_G$ with the obtained formula we get the required result. ■

## Acknowledgments

The work is supported by the Mathematical Center in Akademgorodok (Novosibirsk).

## References


1. Klein D.J., Randić M. Resistance distance // Journal of Mathematical Chemistry. 1993. Vol. 12, no. 1-4. P. 81–95.
2. Gutman I., Mohar B. The quasi-Wiener and the Kirchhoff indices coincide // Journal of Chemical Information & Computer Sciences. 1996. Vol. 36. P. 982–985.
3. Zhu H.Y., Klein D.J., Lukovits I. Extensions of the Wiener number // Journal of Chemical Information and Modeling. 1996. Vol. 36, no. 3. P. 420–428.
4. Zhang H., Yang Y. Resistance distance and Kirchhoff index in circulant graphs // International Journal of Quantum Chemistry. 2007. Vol. 107, no. 2. P. 330–339.
5. Luzhen Y. On the Kirchhoff index of some toroidal lattices // Linear and Multilinear Algebra. 2011. Vol. 59, no. 6. P. 645–650.
6. Cinkir Z. Effective resistances and Kirchhoff index of ladder graphs // Journal of Mathematical Chemistry. 2016. Vol. 54, no. 4. P. 955–966.
7. Baigonakova G.A., Mednykh A.D. Elementary formulas for Kirchhoff index of Möbius ladder and prism graphs // Siberian Electronic Mathematical Reports. 2019. Vol. 16. P. 1654–1661.
8. Mednykh A.D., Mednykh I.A. Kirchhoff index for circulant graphs and its asymptotics // Siberian Mathematical Journal. 2024. Vol. 65, no. 6, P. 1359–1372.
9. Davis P.J. Circulant Matrices. New York: AMS Chelsea Publishing, 1994.
10. Mason J.C., Handscomb D.C. Chebyshev Polynomials. New York: CRC Press. Boca Raton, 2003.

# ОБ ИНДЕКСЕ КИРХГОФФА ДЛЯ ЦИРКУЛЯНТНОГО ГРАФА С НЕФИКСИРОВАННЫМИ СКАЧКАМИ

**Г.К. Соколова**[1,2]
аспирант, ассистент, e-mail: g.sokolova@g.nsu.ru

[1]Новосибирский государственный университет, Новосибирск, Россия
[2]Новосибирский государственный технический университет, Новосибирск, Россия

**Аннотация.** Рассматривается инвариант графа, называемый индексом Кирхгоффа. Определяется явная аналитическая формула индекса Кирхгоффа для циркулянтных графов с нефиксированными скачками.

**Ключевые слова:** циркулянтный граф с нефиксированными скачками, собственные значения Лапласиана графа, индекс Кирхгоффа для графа.

*Дата поступления в редакцию: 11.03.2025*