# TOWARDS OPTIMAL TECHNIQUES INTERMEDIATE BETWEEN INTERVAL AND AFFINE, AFFINE AND TAYLOR

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**Abstract.** In data processing, it is important to gauge how input uncertainty affects the results of data processing. Several techniques have been proposed for this gauging, from interval to affine to Taylor techniques. Some of these techniques result in more accurate estimates but require longer computation time, others' results are less accurate but can be obtained faster. Sometimes, we do not have enough time to use more accurate (but more time-consuming) techniques, but we have more time than needed for less accurate ones. In such cases, it is desirable to come up with intermediate techniques that would utilize the available additional time to get somewhat more accurate estimates. In this paper, we formulate the problem of selecting the best intermediate techniques, and provide a solution to this optimization problem.

**Keywords:** data processing, interval method, optimal method, Taylor method.

## 1.  Formulation of the Problem

**Interval, affine, and Taylor techniques: reminder.** In many practical problems, we need to estimate the value of a quantity $y$ based on the values of the quantities $x_1, \ldots, x_n$ on which $y$ depends in a known way, as $y = f(x_1, \ldots, x_n)$ for a known algorithm $f(x_1, \ldots, x_n)$.

The problem is that we do not know the exact values of the quantities $x_i$, all we know are the results $\widetilde{x}_i$ of measuring $x_i$, and these results are, in general different from the actual values of the corresponding quantities: there is usually a non-zero measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$; see, e.g., [7]. Often, the only information that we have about each of these measurement error is the upper bound $\Delta_i$ on its absolute value: $|\Delta x_i| \leqslant \Delta_i$. In this case, the only information that we have about the actual (unknown) value $x_i$ is that this value belongs to the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$. In such situations, it is desirable not only to compute the value $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, but also to find the range of possible values of $y$:

$$\{f(x_1, \ldots, x_n) : x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]\}.$$

One of the natural ideas for computing this range is to take into account that computing $y$ consists of several computational steps. So, on each of these steps, we do not only compute the corresponding intermediate result $z$, but we also keep some information about the dependence of this result on $x_i$, information that will eventually help us to find the desired range. There exist several implementations of this idea.

- In interval computations (see, e.g., [4–6]), for each intermediate result $z$, we keep an interval of possible values of $z$.

- In affine arithmetic (see, e.g., [2, 3]), for each intermediate result $z$, we represent $\Delta z = \widetilde{z} - z$ as the expression

$$\Delta z = \sum_{i=1}^{n} a_i \cdot \Delta x_i + \delta z,$$

  in which we know the coefficients $a_i$ and the upper bound $\Delta_z$ on the absolute value of the remaining term $\delta z$: $|\delta z| \leqslant \Delta_z$.

- In the more general Taylor arithmetic (see, e.g., [1]), instead of a generic linear expression, we keep a generic polynomial expression of a given order $k$:

$$\Delta z = \sum_{i_1=1}^{n} a_{i_1} \cdot \Delta x_{i_1} + \ldots + \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} a_{i_1 \ldots i_k} \cdot \Delta x_{i_1} \cdot \ldots \cdot \Delta x_{i_k} + \delta_z,$$

  in which we know the coefficients $a_{i_1 \ldots i_j}$ and the upper bound $\Delta_z$ on the absolute value of the remaining term $\delta z$.

Then, for each elementary computational step – addition, subtraction, multiplication, etc. – we use expressions for this step's inputs to come up with a similar expression for the output of this step. For example, if we know that

$$\Delta z = \sum_{i=1}^{n} a_i \cdot \Delta x_i + \delta z \;\; and \;\; \Delta t = \sum_{i=1}^{n} b_i \cdot \Delta x_i + \delta t,$$

with $|\delta z| \leqslant \Delta_z$ and $|\delta t| \leqslant \Delta_t$, then for $s = z + t$, we get

$$\Delta s = \sum_{i=1}^{n} (a_i + b_i) \cdot \Delta x_i + \delta s,$$

where $|\delta s| \leqslant \Delta_z + \Delta_s$.

**Need for intermediate techniques.** The more terms we keep in the dependence of $\Delta z$ on $\Delta x_i$, the more accurately we represent this dependence – after all, any continuous function on a bounded domain can be approximated by polynomials as accurately as possible, but the more accuracy we want, the more terms we need. On the other hand, the more terms we keep and process for each intermediate

result, the more memory we need and the more computation time we need – and both memory and computation time are often limited.

As of now, the usual choice is either go with interval computations, or use affine arithmetic, or use quadratic Taylor series, or use cubic Taylor series, etc. But what if we do not have enough time to use affine techniques but we still have extra time left when using intervals? In this case, it is desirable to use this extra time to come up with computations which are less time consuming that affine arithmetic, but more accurate than interval computations. Similarly, if we cannot afford quadratic Taylor series but we still have extra time left when using affine arithmetic, it is desirable to come up with computations which are less time consuming that quadratic Taylor technique, but more accurate than affine arithmetic.

**Which intermediate techniques should we choose?** There are many possible intermediate techniques. We can choose some monomials and only use their linear combinations. Alternatively, we can select some other basis in the linear space of all polynomials of given order, and use linear combinations of some elements of this basis.

In this paper, we show that the optimal choice is selecting monomials.

## 2. Analysis of the Problem

**What we want.** If we can only afford to have a limited number $L$ of coefficients at each computation stage, then we need to represent the difference $\Delta z$ corresponding to each intermediate result as

$$\Delta z = \sum_{\ell=1}^{L} a_\ell \cdot f_\ell(\Delta x_1, \ldots, \Delta x_n) + \delta z,$$

where $f_\ell(\Delta x_1, \ldots, \Delta x_n)$ are pre-selected analytical functions, and we know the coefficients $a_\ell$ and a bound $\Delta_z$ of the absolute value of the remainder $\delta z$.

In this approach, we approximate each dependence of $\Delta z$ on $\Delta x_i$ by a linear combination of the functions $f_\ell(\Delta x_1, \ldots, \Delta x_n)$, i.e., by an element of the corresponding $L$-dimensional space

$$A = \left\{ \sum_{\ell=1}^{L} a_\ell \cdot f_\ell(\Delta x_1, \ldots, \Delta x_n) \right\}_{a_1, \ldots, a_L}.$$

So, selecting an intermediate method means selecting an $L$-dimensional linear (sub)space in the linear space of all analytical functions.

**What we mean by optimal.** We want to select a subspace which is, in some reasonable sense, optimal. In some cases, optimal means attaining the largest or the smallest value of some objective function. However, optimality criteria can be more general. For example, if we select average approximation error as the objective function, we may end up with several different spaces with the same smallest possible value of this objective function. In this case, it is reasonable to select,

among them, the space that requires the smallest possible average computation time. This is equivalent to selecting an optimality criterion which is more complex than numerical: according to this criterion, a space $A$ is better than a family $A'$ if:

- either $A$ has a smaller average approximation error,

- or they have the same average approximation error, but $A'$ has a smaller average computation time.

We can have even more complex criteria. In general, what all these criteria do is for some pairs of alternatives $A$ and $A'$ that $A$ is better – we will denote it by $A < A'$ – or that they are of equal quality with respect to this criterion; this we denote by $A \sim A'$. It is also possible that for some pairs, the criterion does not tell us which alternative is worse. Of course, these conclusions should be consistent: e.g., if $A$ is better than $A'$, and $A'$ is better than $A''$, then $A$ should be better than $A''$.

What is important is that there should be exactly one alternative which is, according to this criterion, better than or of equal quality than all others. Indeed, as we have mentioned, if there are several optimal alternatives, this would mean that we can use the corresponding non-uniqueness to optimize something else – and thus, that the original optimality criterion is not final.

**Scale-invariance.** We process the values of physical quantities, but the numerical values of these quantities depend on the choice of a measuring unit. If we replace meters with centimeters, the lengths remain the same, but the numerical values of all the lengths become multiplied by $c = 100$. In general, if we select a different measuring unit for the quantity $x_i$, then its numerical value (and thus, the numerical value of the difference $\Delta x_i = \widetilde{x}_i - x_i$) gets multiplied by the corresponding factor $c_i > 0$: $x_i \mapsto c_i \cdot x_i$.

It is reasonable to assume that the relative quality of different approximation families do not depend on the choice of units. Indeed, it would be very strange if one family is better for meters and kilograms, and another is better for centimeters and grams.

Now, we are ready to formulate our main result.

## 3.  Definition and the Main Result

**Definition 1.** *Let $\mathcal{A}$ be a set; its elements will be called alternatives.*

- *By an* optimality criterion *on the set $\mathcal{S}$, we mean a pair of relations $(<, \sim)$ that satisfy the following properties:*

  - *if $A < A'$ and $A' < A''$, then $A < A''$;*
  - *if $A < A'$ and $A' \sim A''$, then $A < A''$;*
  - *if $A \sim A'$ and $A' < A''$, then $A < A''$;*
  - *if $A \sim A'$ and $A' \sim A''$, then $A \sim A''$;*

– *always $A \sim A$; and*

– *if $A < A'$ then $A \not\sim A'$ and $A' \not< A$.*

- *We say that an alternative $A$ is* optimal *for every $A' \in \mathcal{A}$, we have $A < A'$ or $A \sim A'$.*

- *We say that an optimality criterion is* final *if there is exactly one optimal alternative.*

**Definition 2.** *Let $\mathcal{A}$ be the set of all $L$-dimensional linear subspaces of the linear space of all analytical functions. We say that the optimality criterion is* scale-invariant *if for all tuples $c = (c_1, \ldots, c_n)$ of positive numbers, we have $A < A' \Leftrightarrow S_c(A) < S_c(A')$ and $A \sim A' \Leftrightarrow S_c(A) \sim S_c(A')$, where*

$$S_c(A) = \{f(c_1 \cdot \Delta x_1, \ldots, c_n \cdot \Delta x_n) : f(\Delta x_1, \ldots, \Delta x_n) \in A\}.$$

**Proposition.** *For every scale-invariant final optimality criterion, the optimal linear space is the set of all linear combinations of given $L$ monomials.*

*Comment.* In other words, the optimal method between interval and affine means selecting $L < n$ variables $i_1, \ldots, i_L$, and considering expressions

$$\Delta z = \sum_{\ell=1}^{L} a_\ell \cdot x_{i_\ell} + \delta z.$$

The optimal method between affine and quadratic Taylor methods means selecting $L - n$ pairs $(i_\ell, j_\ell)$, and considering expressions

$$\Delta z = \sum_{i=1}^{n} a_i \cdot x_i + \sum_{\ell=1}^{L-n} a_\ell \cdot x_{i_\ell} \cdot x_{j_\ell} + \delta z,$$

etc.

**Proof of the Proposition.** Let us first prove that the optimal space $A_{\mathrm{opt}}$ is itself scale-invariant, i.e., that $T_c(A_{\mathrm{opt}}) = A_{\mathrm{opt}}$ for all $c$. Indeed, by definition of optimality, for every $A'$, we have $A_{\mathrm{opt}} < A'$ or $A_{\mathrm{opt}} \sim A'$. This is true for all $A'$, in particular, for $A' = T_{c^{-1}}(A)$, where $(c_1, \ldots, c_n)^{-1} \stackrel{\text{def}}{=} (c_1^{-1}, \ldots, c_n^{-1})$. By using scale-invariance, from $A_{\mathrm{opt}} < T_{c^{-1}}(A)$, we conclude that $T_c(A_{\mathrm{opt}}) < T_c(T_{c^{-1}}(A)) = A$, and from $A_{\mathrm{opt}} \sim T_{c^{-1}}(A)$, we conclude that $T_c(A_{\mathrm{opt}}) \sim T_c(T_{c^{-1}}(A)) = A$. Thus, for each alternative $A$, we have either $T_c(A_{\mathrm{opt}}) < A$ or $T_c(A_{\mathrm{opt}}) \sim A$. By definition of an optimal alternative, this means that the alternative $T_c(A_{\mathrm{opt}})$ is optimal. But our optimality criterion is final, which means that there is only one optimal alternative, and therefore, $T_c(A_{\mathrm{opt}}) = A_{\mathrm{opt}}$.

Each function from the basis of the optimal family is an analytical function, i.e., a sum – finite or infinite – of monomials, i.e., of the expressions of the type

$(\Delta x_1)^{k_1} \cdot \ldots \cdot (\Delta x_n)^{k_n}$. Let $m_1$ be the smallest possible value of $k_1$ in all $L$ basic functions. Then, the function containing a non-zero term with $x_1^{m_1}$ has the form

$$f_\ell(\Delta x_1, \Delta_2, \ldots, \Delta x_n) =$$

$$(\Delta x_1)^{m_1} \cdot P_1(\Delta x_2, \ldots, \Delta x_n) + (\Delta x_1)^{m_1+1} \cdot P_2(\Delta x_2, \ldots, \Delta x_n) + \ldots,$$

where $P_k$ are polynomials and $P_1$ is not identically 0. Due to scale-invariance, for each $c_1$, the function

$$f_\ell(c_1 \cdot \Delta x_1, \Delta_2, \ldots, \Delta x_n) =$$

$$c_1^m \cdot (\Delta x_1)^{m_1} \cdot P_1(\Delta x_2, \ldots, \Delta x_n) + c_1^{m+1} \cdot (\Delta x_1)^{m_1+1} \cdot P_2(\Delta x_2, \ldots, \Delta x_n) + \ldots$$

also belongs to the space $A_{\mathrm{opt}}$, and thus, the function

$$c_1^{-m} \cdot f_\ell(c_1 \cdot \Delta x_1, \Delta_2, \ldots, \Delta x_n) =$$

$$(\Delta x_1)^{m_1} \cdot P_1(\Delta x_2, \ldots, \Delta x_n) + c_1 \cdot (\Delta x_1)^{m_1+1} \cdot P_2(\Delta x_2, \ldots, \Delta x_n) + \ldots$$

A finite-dimensional linear space is closed, i.e., contains all its limits. In particular, in the limit $c_1 \to 0$, we conclude that the space $L$ contains the function

$$(\Delta x_1)^{m_1} \cdot P_1(\Delta x_2, \ldots, \Delta x_n).$$

Similarly, by considering the smallest possible power of $\Delta x_2$ in this expression and using scale-invariance, we conclude that the optimal linear space contains a function $(\Delta x_1)^{m_1} \cdot (\Delta x_2)^{m_2} \cdot Q_1(\Delta x_3, \ldots, \Delta x_n)$, etc., and in the end, that the optimal linear space contains a monomial $(\Delta x_1)^{m_1} \cdot (\Delta x_2)^{m_2} \cdot \ldots \cdot (\Delta x_n)^{m_n}$.

By subtracting terms proportional to this monomial from all the basic functions, we thus get a new basis, in which we can also select a monomial, etc. At the end, we indeed get a representation of the optimal linear space as the set of all linear combinations of $L$ monomials.

The proposition is proven.

**How we can implement this idea.** In the case of techniques intermediate between interval and affine, we can select the variables $x_i$ for which the initial uncertainty is the largest.

Alternatively, at each step like computing $s = z + t$, we can first combine all $2L$ terms from both expressions for $z$ and for $t$, and then keep $L$ of them with the largest uncertainty – i.e., the largest values of the corresponding term $|a_i| \cdot \Delta_i$.

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## НА ПУТИ К ОПТИМАЛЬНЫМ МЕТОДАМ, ПРОМЕЖУТОЧНЫМ МЕЖДУ ИНТЕРВАЛЬНЫМ И АФФИННЫМ, АФФИННЫМ И ТЕЙЛОРОВСКИМ

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**Аннотация.** При обработке данных важно оценить, как неопределенность ввода влияет на результаты обработки данных. Для этой калибровки было предложено несколько методов, от интервальных до аффинных и методов Тейлора. Некоторые из этих методов дают более точные оценки, но требуют больше времени для вычислений, другие результаты менее точны, но могут быть получены быстрее. Иногда у нас не хватает времени, чтобы использовать более точные (но более трудоёмкие) методы, но у нас есть больше времени, чем нужно, для менее точных. В таких случаях желательно разработать промежуточные методы, которые позволили бы использовать имеющееся дополнительное время для получения несколько более точных оценок. В данной статье мы формулируем задачу выбора наилучших промежуточных методов и даём решение этой задачи оптимизации.

**Ключевые слова:** обработка данных, интервальный метод, оптимальный метод, метод Тейлора.