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WHY T-DUALITY: A SIMPLE EXPLANATION

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Abstract. In many physical theories, there is a — somewhat surprising — similarity between events corresponding to large distances R and events corresponding to very small distances $1/R$. Such similarity is known as T-duality. At present, the only available explanation for T-duality comes from a complex mathematical analysis of the corresponding formulas. In this paper, we provide an alternative explanation based on the fundamental notion of causality.

Keywords: T-duality, causality relation, causality-preserving transformations, inversion.

1. Formulation of the Problem

What is T-duality. In many physical theories, there is a strong similarity between effects at large distances R and at small distances $1/R$. This similarity is known as *T-duality*; see, e.g., [18, 20, 21].

T-duality relates two areas of physics which are among the most difficult to study (and thus, the most mysterious): the study of very large objects (of cosmological size) and the study of very small objects (of size below the usual particle size).

Comment. The term T-duality is sometimes also used to describe a general similarity between two physical theories.

Need for a simple explanation. At present, T-duality arrives via a complex analysis of the corresponding mathematical models.

It would be beneficial to come up with a simple — and more fundamental — explanation, an explanation that would not depend on the specific complex mathematical details (which may change as theories evolve), but that would be based on fundamental physical ideas.

What we do in this paper. In this paper, we provide such a simple and fundamental explanation of T-duality.

2. Our Explanation

Causality is one of the most fundamental physical phenomenon. We are interested in the explanation based on fundamental physical concepts. One of the most fundamental physical concept is the concept of *causality*: the idea that some events in space-time can influence each other; see, e.g., [8, 22].

The fundamental character of causality implies that for a transformation to preserve physical properties, this transformation should also preserve causality. Let us therefore recall which space-time transformations preserve the causality relation.

For this analysis, we need to recall how causality is described in modern physics.

How is causality described on the local level. According to modern physics, locally — i.e., in some vicinity of an event — metric is close to Minkowski one, and the causality relation $a \leqslant b$ between two space-time events $a = (a_0, a_1, \ldots, a_n)$ and $b = (b_0, b_1, \ldots, b_n)$ is described by the formula

$$a \leqslant b \leftrightarrow a = b \vee (b_0 \geqslant a_0 \,\&\, (b-a)^2 \geqslant 0),$$

where n is the dimension of proper space and $a^2 \stackrel{\text{def}}{=} a_0^2 - a_1^2 - \ldots - a_n^2$.

Comment. Here, for simplicity, we assume that time and distance are measured in the same units, i.e., that the units are selected in such a way that the speed of light c is equal to 1. If we use different units for measuring space and time, then we will have $a^2 = c^2 \cdot a_0^2 - a_1^2 - \ldots - a_n^2$.

What transformations preserve causality: ideal case. Let us start with an ideal case, in which the causality relation in the whole space-time E is described by the above Minkowki causality relation.

It is known that for every $n \geqslant 2$, every bijection $E \to E$ which preserves the Minkowski causality relation is linear; moreover, it is a composition of Lorentz transformations, shirts, rotations, and dilations.

To be precise:

- A *Lorentz transformation* is a mapping

  $$(a_0, \vec{a}) \to \left(\frac{a_0 - \vec{v} \cdot \vec{a}}{1 - \vec{v} \cdot \vec{v}}, \frac{\vec{a} - a_0 \cdot \vec{v}}{1 - \vec{v} \cdot \vec{v}}\right),$$

  where $\vec{v} \cdot \vec{a} \stackrel{\text{def}}{=} v_1 \cdot a_1 + \ldots + v_n \cdot a_n$, and $\vec{v} \cdot \vec{v} \leqslant 1$.

- A *rotation* is a mapping $(a_0, \vec{a}) \to (a_0, Ta)$, where T is a rotation in the n-dimensional Euclidean space.

- A *shift* is a mapping $a \to a + b$, for some $b \in E$.

- A *dilation* is a mapping $a \to \lambda \cdot a$, for some real number λ.

This theorem was first proven by A. D. Alexandrov [1, 5]; see also [2, 3, 6, 7, 9, 10, 12–17, 19, 23].

Towards a more realistic case. As we have mentioned earlier, Minkowski causality is only a local approximation. So, a natural question is: what are transformations that preserve Minkowski causality in a bounded domain?

The answer to this question was also provided by A. D. Alexandrov; see, e.g., [4, 9]. For bounded domains, in addition to linear transformations, we also have special nonlinear transformations — inversions:

- An *inversion* is a mapping $a \to \dfrac{a-b}{(a-b)^2} + b$, for some $b \in E$.
- A *singular double inversion* is a mapping

  $$a \to \frac{(a-b) + c \cdot (a-b)^2}{1 + 2 \cdot c(a-b)} + b$$

  for some $b \in E$ and $c \in E$ for which $c^2 = 0$.
- By a *conformal mapping*, we mean one of the above transformations or their composition.

Alexandrov's result is that every bijection $f : D \to D$ of a bounded domain D that preserves Minkowski causality is a conformal mapping.

Towards an even more realistic case. The actual causality relation is only approximately described by the Monkowski formula: the smaller the neighborhood, the closer we are to the Minkowski causality. How can we describe transformations that preserve approximately-Minkowski causality?

Such transformations were described in [11]: it turns out that for causality relations which are sufficiently close to the Minkowski ones, the transformations that preserve (or at least approximately preserve) causality are close to conformal mappings.

This leads to the desired explanation of T-duality. Indeed, here, in addition to the usual Minkowski transformations, we also have inversions — which correspond exactly to transformations $R \to 1/R$ (plus dilations), and double inversions — which can be viewed as limits of compositions of two consecutive inversions.

Thus, indeed, T-duality naturally follows from the fundamental notion of causality.

This also explains why T-duality is approximate. The above arguments also explain why T-duality is not an exact equivalence — i.e., why there is a difference between cosmological and micro-world laws: inversions preserve causality, but they do not preserve the actual metric.

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References

1. Alexandrov A.D. On Lorentz transformations. Uspekhi Math. Nauk, 1950, vol. 5, no. 3(37), pp. 187. (in Russian)
2. Alexandrov A.D. A contribution to chronogeometry. Canadian Journal of Mathematics, 1967, vol. 19, pp. 1119–1128.
3. Alexandrov A.D. Mappings of spaces with families of cones and space-time transformations. Annali di Matematica Pura ed Aplicata, 1967, vol. 103, pp. 229–257.
4. Alexandrov A.D. On the axioms of relativity theory. Vestnik Leningrad University, Ser. Math., 1976, no. 19, pp. 5–28.
5. Alexandrov A.D. and Ovchinnikova V.V. Remarks on the foundations of special relativity. Leningrad University Vestnik, 1953, no. 11, pp. 94–110. (in Russian)
6. Benz W. A characterization of plane Lorentz transformations. Journal of Geometry, 1977, vol. 10, pp. 45–56.
7. Benz W. Geometrische Transformationen. BI Wissenscahftsverlag, Mannheim, 1992.
8. Feynman R., Leighton R., and Sands M. The Feynman Lectures on Physics. Addison Wesley, Boston, Massachusetts, 2005.
9. Guts A.K. Axiomatic relativity theory. Russian Math. Surveys, 1982, vol. 37, no. 2, pp. 41–89.
10. Kosheleva O.M., Kreinovich V., and Vroegindewey P.G. An extension of a theorem of A. D. Alexandrov to a class of partially ordered fields. Proceedings of the Royal Academy of Science of Netherlands, 1979, vol. 82(3), series A, September 21, pp. 363–376.
11. Kreinovich V. and Kosheleva O. From (idealized) exact causality-preserving transformations to practically useful approximately-preserving ones: a general approach. International Journal of Theoretical Physics, 2008, vol. 47, no. 4, pp. 1083–1091.
12. Kuzminykh A.V. Characterization of Lorentz transformations. Soviet Math. Dokl., 1975, vol. 16, no. 6, pp. 1626–1628.
13. Kuzminykh A.V. Minimal condition determining the Lorentz transformations. Siberian Mathematical Journal, 1976, vol. 17. no. 6, pp. 968–972.
14. Lester J.A. On null cone preserving mappings. Proceedings of Cambridge Mathematical Society, 1977, vol. 81, pp. 455–462.
15. Lester J.A. Cone preserving mappings for quadratic cones over arbitrary fields. Canadian Journal of Mathematics, 1977, vol. 29, pp. 1247–1253.
16. Lester J.A. A physical characterization of conformal transformations of Minkowski spacetime. Annals of Discrete Mathematics, 1983, vol. 18, pp. 567–574.
17. Lester J.A. Distance preserving transformations. In: F. Buekenhout (ed.), Handbook of Incidence Geometry, Elsevier, Amsterdam, 1995.
18. Mack K. The End of Everything (Astronomically Speaking). Scribner, New York, 2020.
19. Naber G.L. The geometry of Minkowski space-time. Springer-Verlag, N.Y., 1992.
20. Sathiapalan B. Duality in statistical mechanics and string theory. Physical Review Letters, 1987, vol. 58, no. 16, pp. 1597–1599.
21. Seiberg N. Emergent spacetime. In: D. Gross, M. Henneaux, and A. Sevrin (eds.), The Quantum Structure of Space and Time: Proceedings of the 23rd Solvay Conference on Physics, Brussels, Belgium, December 1–3, 2005, World Scientific, Singapore, 2007, pp. 163–213.

22. Thorne K.S. and Blandford R.D. Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics. Princeton University Press, Princeton, New Jersey, 2017.
23. Zeeman E.C. Causality implies the Lorentz group. Journal of Mathematical Physics, 1964, vol. 5, pp. 490–493.

ПРОСТОЕ ОБЪЯСНЕНИЕ Т-ДУАЛЬНОСТИ

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Аннотация. Во многих физических теориях существует несколько удивительное сходство между событиями, соответствующими большим расстояниям R и очень малым расстояниям 1/R. Такое подобие известно как Т-дуальность. В настоящее время единственное доступное объяснение Т-дуальности следует из сложного математического анализа соответствующих формул. В этой статье мы предлагаем альтернативное объяснение, основанное на фундаментальном понятии причинности.

Ключевые слова: Т-дуальность, причинно-следственная связь, преобразования, сохраняющие причинность, инверсия.

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