TOWARDS A PHYSICS-MOTIVATED SMALL-VELOCITIES APPROXIMATION TO GENERAL RELATIVITY

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Abstract. In the general case, complex non-linear partial differential equations of General Relativity are very hard to solve. Thus, to solve the corresponding physical problems, usually appropriate approximations are used. The first approximation to General Relativity is, of course, Newton's theory of gravitation. Newton's theory is applicable when the gravitational field is weak and when all velocities are much smaller than the speed of light. Most existing approximations allow higher velocities, but still limit us to weak gravitational fields. In this paper, he consider the possibility of a different approximation, in which strong fields are allowed but velocities are required to be small. We derive the corresponding equations and speculate on their possible physical consequences.

Keywords: General Relativity, small-velocities case, approximation.

1. Formulation of the Problem

Need for approximations to General Relativity. Since its discovery in 1915 and its first experimental confirmation in 1919, General Relativity has been consistently confirmed by more and more accurate experiments [1, 6, 10, 11]. However, its non-linear differential equations are very difficult to solve in the general case. As a result, most observable physical consequences – starting from Einstein's own – were obtained by using appropriate approximations to General Relativity.

In many cases, when a new problem appeared which could not be easily solved by using the existing approximation, a solution was found when a new approximation was developed. Since General Relativity still has many problems which have not yet been fully analyzed, a reasonable idea is to come up with new approximations, with the hope that these approximations will help us to analyze these problems.

It is desirable to come up with physics-motivated approximations. Sometimes, purely mathematical and/or computational approximation ideas are very helpful. However, the experience of physicists shows that, in general, physics-motivated approximation are usually much more successful [1]. From this view-

point, it is desirable to come up with new physics-motivated approximation to General Relativity.

Which approximations are known. General Relativity is, in its essence, the theory of gravitational interactions. From this viewpoint, it is an extension of the original Newton's theory of gravitation. In this sense, the Newton's theory is the first approximation to General Relativity.

Newton's theory adequately describes gravitational effects when the following two conditions are satisfied:

- the gravitational field φ is weak and

- all the velocities v are much smaller than the speed of light c: $v \ll c$.

In the vicinity of the Solar system, the gravitational field is weak, but there are objects – such a light – which travel with the speed which is equal to the speed of light. As a result, while the Newtonian theory of gravitation provides a very accurate description of the movement of celestial bodies, it is not so adequate in describing the gravitational effects on the objects whose velocities are comparable with the speed of light. For example, the Newtonian estimate of the Sun's gravitational effect on the trajectory of light is twice smaller than what General Relativity predicts, and the observational confirmation of this effect in 1919 was the first of many successes of this theory.

From this viewpoint, it has been reasonable to look for a next approximation in which we take into account that the velocities may be comparable with the speed of light. Such a *post-Newtonian* approximation was indeed developed. This effort was started by Einstein himself. This approximation was later supplemented by taking into account not only terms which are *linear* in terms of the (weak) gravity field (as Newton's theory does), but also terms which are *quadratic* in terms of this field [6, 10, 11].

Possibility of an alternative approximation: what we do in this paper. Newton's theory holds under the two above assumptions: that the field is weak and that the velocities are small. Current approximations largely describe what happens when we do not make the second assumption, while keeping the first assumption largely intact.

From the physical viewpoint, it is also natural to try to analyze what will happen if we do not make the first assumption (about weak fields), while keeping the second one (that velocities are small) intact. In this paper, we show what approximation we get when we try to come up with such a small-velocity approximation.

Comment. Some of the results presented in this paper first appeared in the technical report [8].

2.   Analysis of the Problem

To come up with a physics-motivated approximation, let us recall the physics behind these equations. To come with a physics-motivated approximation, let us

start not directly with the equations of General Relativity, but first with the physics behind these equations.

Einstein himself came up with the equations of General Relativity from physical ideas like equivalence principle, but he used these ideas more as a heuristic, he did not show how these ideas lead exactly to his equations and not to any other equations; such derivations only appeared much later (see, e.g., [2, 3]).

Historically the first physics-motivated derivation of Einstein's equation was obtained by Gupta [4, 5] (see also [7]) based on a slightly different idea. Specifically, Gupta used another idea behind General Relativity: that the source of the gravitational field g_{ij} is the *total* energy density, including both:

- the energy of the other fields, *and*

- the energy of the gravitational field itself.

The fact that a gravitational field generated by other bodies and fields itself becomes a source of gravity explains the highly non-linear character of Einstein's equations. This idea is what we will use to come up with our small-velocities approximation.

Comment. It should be mentioned that while this idea can be traced back to Einstein himself, it was only in 1954, after the first Gupta's paper, that it was proven that this idea explains the exact equations of General Relativity.

How to describe a small-velocities approximation: a general reminder. In General Relativity, gravitation is described by the metric tensor g_{ij}, so that the proper times ds between the space-time points x^i and $x^i + dx^i$ has the form $ds^2 = \sum_i \sum_j g_{ij} \cdot dx^i \cdot dx^j$. Here, $dx^0 = dt$ is the difference in time, and $dx^a = v_a \cdot dt$ for $a = 1, 2, 3$, where v_a is the a-th component of the velocity v. Thus,

$$ds^2 = dt^2 \cdot \left(g_{00} + 2 \sum_a g_{0a} \cdot v_a + \sum_a \sum_b g_{ab} \cdot v_a \cdot v_b \right).$$

In the small-velocities approximation, when $v_a \ll c$, we can ignore terms proportional to these small velocities, and thus, consider only the g_{00} component of the gravitational field [1, 10].

So, in this approximation, gravity at each space-time point is described by a single scalar g_{00}. In other words, gravity is described by a single scalar field. For simplicity, let us denote this field by φ.

A general description of physical fields in a small-velocity approximation is well known in classical field theory [1, 9]. Such theories are usually formulated in terms of the corresponding minimum action principle $S = \int L \, dx \to \min$, where the corresponding Lagrangian has the form $L = L(\varphi, \varphi_{,a} \cdot \varphi_{,a})$, where, as usual, $\varphi_{,a}$ denotes the partial derivative $\dfrac{\partial \varphi}{\partial x_a}$, and it is implicitly assumed that we sum over repeated indices, so that, e.g., $\varphi_{,a} \cdot \varphi_{,a}$ means $\sum_{a=1}^{3} \varphi_{,a} \cdot \varphi_{,a}$.

In particular, the original Newton's theory corresponds to the Lagrangian

$$L = k \cdot \rho \cdot \varphi - \frac{1}{2} \cdot \varphi_{,a} \cdot \varphi_{,a},$$

where ρ is the overall energy density of all the other fields, and k is the parameter in the differential equations that describe Newton's theory:

$$\nabla^2 \varphi = -k \cdot \rho. \tag{1}$$

Comment. This small-velocity approximation is what is usually called a *non-relativistic* approximation in field theory, but we will try to avoid this term, since its use of the word "relativistic" (meaning here related to Special Relativity) may be confusing in this text, where relativity means General Relativity.

How to derive equations and energy density from the Lagrangian: a brief reminder. Minimization principle means that the (variational) derivative off the Lagrangian L with respect to the field φ should be equal to 0: $\dfrac{\delta L}{\delta \varphi} = 0$. It is known [1, 9] that this leads to the following partial differential equation:

$$\frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x_a} \left(\frac{\partial L}{\partial \varphi_{,a}} \right) = 0. \tag{2}$$

The energy density of a field with Lagrangian L is described by the formula

$$\rho_L = -L + \varphi_{,a} \frac{\partial L}{\partial \varphi_{,a}}. \tag{3}$$

Now, we are ready to formulate our main idea in precise terms. We want to describe a field whose source includes both the outside energy density ρ and the energy density ρ_L of the field itself, i.e., a field for which

$$\nabla^2 \varphi = -k \cdot \rho - k \cdot \rho_L. \tag{4}$$

Substituting the expression (3) into the formula (4), we get

$$\nabla^2 \varphi = -k \cdot \rho - k \cdot \left[-L + \varphi_{,a} \frac{\partial L}{\partial \varphi_a} \right]. \tag{5}$$

So, *we are looking for a Lagrangian L for which the field equations (2) are equivalent to the equation (5).*

3. Deriving the Approximate Lagrangian

Let us fix a reasonable class of Lagrangians. To solve our problem, let us consider the following natural generalization of the above Newtonian Lagrangian, namely, a Lagrangian of the type

$$L = a \cdot \rho \cdot a(\varphi) - \frac{1}{2} \cdot b(\varphi) \cdot \varphi_{,a} \cdot \varphi_{,a} \tag{6}$$

for appropriate functions $a(\varphi)$ and $b(\varphi)$.

What our idea means for a Lagrangian of this type. Substituting the expression (6) into the field equations (2), we conclude that

$$k \cdot a' \cdot \rho - \frac{1}{2} \cdot b' \cdot \varphi_{,a} \cdot \varphi_{,a} + \partial_a(b \cdot \varphi_{,a}) = 0, \tag{6}$$

where a' and b', as usual, denote the derivatives, and ∂_a denotes partial derivative relative to x_a.

Differentiating the product $b \cdot \varphi_{,a}$, we get

$$k \cdot a' \cdot \rho - \frac{1}{2} \cdot b' \cdot \varphi_{,a} \cdot \varphi_{,a} + b' \cdot \varphi_{,a} \cdot \varphi_{,a} + b \cdot \nabla^2 \varphi = 0, \tag{7}$$

i.e., equivalently, that

$$\nabla^2 \varphi = -k \cdot \frac{a'}{b} \cdot \rho + \frac{1}{2} \cdot \frac{b'}{b} \cdot \varphi_{,a} \cdot \varphi_{,a}. \tag{8}$$

On the other hand, substituting the expression (6) into the formula (5), we get

$$\nabla^2 \varphi = -k \cdot \rho - k^2 \cdot a \cdot \rho - \frac{1}{2} \cdot k \cdot b \cdot \varphi_{,a} \cdot \varphi_{,a} + k \cdot b \cdot \varphi_{,a} \cdot \varphi_{,a},$$

i.e.,

$$\nabla^2 \varphi = -k \cdot (1 - k \cdot a) \cdot \rho + \frac{1}{2} \cdot k \cdot b \cdot \varphi_{,a} \cdot \varphi_{,a}. \tag{9}$$

The equations (8) and (9) should be equivalent. Thus, comparing the equations (8) and (9), we conclude that

$$\frac{b'}{b} = k \cdot b \tag{10}$$

and

$$\frac{a'}{b} = 1 - k \cdot a. \tag{11}$$

Finding the dependence $b(\varphi)$. The equation (10) has the form

$$\frac{\frac{db}{d\varphi}}{b} = k \cdot b.$$

Moving all the terms related to b to one side and all terms related to φ to the other side, we get

$$\frac{db}{b^2} = k \cdot d\varphi.$$

Integrating both sides, we get

$$-\frac{1}{b} = k \cdot \varphi + C,$$

so

$$b(\varphi) = \frac{1}{-C - k \cdot \varphi}.$$

When $k \to 0$, we should get the Newtonian Lagrangian, with $b = 1$. Thus, $C = -1$, and the above formula takes the form

$$b(\varphi) = \frac{1}{1 - k \cdot \varphi}. \tag{12}$$

Finding the dependence $a(\varphi)$**.** Substituting this expression into the formula (11), we conclude that

$$\frac{da}{d\varphi} \cdot (1 - k \cdot \varphi) = 1 - k \cdot a.$$

Moving terms containing a and da to one side and terms containing φ and $d\varphi$ to the other side, we conclude that

$$\frac{da}{1 - k \cdot a} = \frac{d\varphi}{1 - k \cdot \varphi}.$$

Multiplying both sides by $-k$, we get

$$\frac{d(1 - k \cdot a)}{1 - k \cdot a} = \frac{d(1 - k \cdot \varphi)}{1 - k \cdot \varphi},$$

so integration leads to

$$\ln(1 - k \cdot a) = \ln(1 - k \cdot \varphi) + c,$$

i.e., to

$$1 - k \cdot a = C \cdot (1 - k \cdot \varphi).$$

When $k \to 0$, we should get the Newtonian term $a(\varphi) = \varphi$. Thus, we conclude that $C = 1$. So, $1 - k \cdot a = 1 - k \cdot \varphi$, and thus,

$$a(\varphi) = \varphi. \tag{13}$$

The resulting Lagrangian. Substituting the expressions (12) and (13) into the formula (6), we get the following Lagrangian

$$L = k \cdot \rho \cdot \varphi - \frac{1}{2} \cdot \frac{\varphi_{,a} \cdot \varphi_{,a}}{1 - k \cdot \varphi}. \tag{14}$$

This is the Lagrangian that we propose as the desired small-velocities approximation to General Relativity.

4. Analyzing the Approximate Lagrangian: First Attempts

What we do in this section. Our main objective is to provide the desired approximation, so that other researchers will analyze this approximation and be able to use it to solve physical problems.

In this section, we start this analysis by considering the simplest possible case of an empty-space solution.

Taking into account that most of the Universe is practically empty. It is well known that, just like most of the Solar system is practically empty – most of the mass is concentrated in a few practically point-wise celestial bodies – same way the Universe is mostly practically empty.

In the areas where $\rho = 0$, the above Lagrangian has a simplified form

$$L = -\frac{1}{2} \cdot \frac{\varphi_{,a} \cdot \varphi_{,a}}{1 - k \cdot \varphi}. \tag{15}$$

Deriving the resulting solution. The Lagrangian (15) can be simplified if we find a function $\Phi(\varphi)$ for which

$$\Phi'(\varphi) = \frac{1}{\sqrt{1 - k \cdot \varphi}}. \tag{16}$$

For this function, the Lagrangian (15) takes the Newtonian form

$$L = -\frac{1}{2} \cdot \Phi_{,a} \cdot \Phi_{,a}. \tag{17}$$

Integrating the formula (16), we conclude that

$$\Phi(\varphi) = -\frac{2}{k} \cdot \sqrt{1 - k \cdot \varphi} + c. \tag{18}$$

When $k \to 0$, we should have $\Phi \approx \varphi$. In this limit, we have $\sqrt{1 - k \cdot \varphi} \approx 1 - \frac{k}{2} \cdot \varphi$, so (18) takes the form

$$\Phi \approx -\frac{2}{k} \cdot \left(1 - \frac{k}{2} \cdot \varphi\right) + c = -\frac{2}{k} + \varphi + c,$$

and the requirement that Φ tend to φ implies that $c = \frac{2}{k}$. Thus, that the formula (18) take the form

$$\Phi = \frac{2}{k} \cdot \left(1 - \sqrt{1 - k \cdot \varphi}\right). \tag{19}$$

For the Lagrangian (17), the variational equations $\frac{\delta L}{\delta \varphi} = 0$ take the usual Newtonian form

$$\nabla^2 \Phi = 0.$$

From Newton's theory, we know the general solution to this equation:

$$\Phi = k \cdot \sum_j \frac{m_j}{r_j},$$
(20)

where m_j is the mass of the j-th body and r_j is the distance to this body.

From (19), we conclude that

$$1 - \sqrt{1 - k \cdot \varphi} = \frac{k}{2} \cdot \Phi,$$

hence

$$\sqrt{1 - k \cdot \varphi} = 1 - \frac{k}{2} \cdot \Phi,$$

$$1 - k \cdot \varphi = \left(1 - \frac{k}{2} \cdot \Phi\right)^2 = 1 - k \cdot \Phi + \frac{k^2}{4} \cdot \Phi^2,$$

and thus,

$$\varphi = \Phi - \frac{k}{4} \cdot \Phi^2.$$
(21)

Conclusion: the resulting solution. The resulting solution has the form (21), where Φ has the form (20), i.e., the form

$$\varphi = k \cdot \sum_j \frac{m_j}{r_j} - \frac{k^2}{4} \cdot \left(\sum_j \frac{m_j}{r_j}\right)^2.$$
(22)

An unexpected consequence: repulsion replaces attraction when the gravitational field becomes very strong. For a gravitational field generated by a single body of mass m, this formula takes the form

$$\varphi = k \cdot \frac{m}{r} - \frac{k^2}{4} \cdot \frac{m^2}{r^2}.$$
(23)

The resulting force is then equal to

$$F = \frac{d\varphi}{dr} = -k \cdot \frac{m}{r^2} + \frac{k^2}{2} \cdot \frac{m^2}{r^3}.$$
(24)

At a distance $r = \dfrac{k \cdot m}{2}$, when the Newtonian field attains a very large value $\dfrac{1}{2}$, the force (24) turns into 0, an for smaller r (i.e., for stronger fields), the force changes sign, i.e., gravitation becomes a repulsion force instead of the usual attraction force.

In physical terms, such a strong field corresponds to a very small-size object of a dis-proportionally huge mass – an object that, according to General Relativity, is most probably a black hole [10].

Acknowledgments

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НА ПУТИ ОТ ФИЗИЧЕСКИ-МОТИВИРОВАННЫХ НИЗКОСКОРОСТНЫХ АППРОКСИМАЦИЙ К ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ

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Аннотация. В общем случае, получить решение сложных нелинейных дифференциальных уравнений общей теории относительности очень трудно. Поэтому для решения соответствующих физических проблем обычно используются подходящие приближения. Первым приближении общей теории относительности является, конечно, теория тяготения Ньютона. Теория Ньютона применима, когда гравитационное поле является слабым и когда все скорости много меньше скорости света. Большинство существующих приближений позволяет работать на более высоких скоростях, но по-прежнему ограничивают нас в слабых гравитационных полях. В данной работе рассматривается возможность получения различных аппроксимаций, которые разрешены в сильных полях, но скорости должны быть небольшие. Мы выведем соответствующие уравнения и обсудим их возможные физические интерпретации.

Ключевые слова: общая теория относительности, случай небольшой скорости, приближение.