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SURFACE CRITICALITY IN RANDOM FIELD SYSTEMS WITH CONTINUOUS SYMMETRY

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We study the surface scaling behavior of a d-dimensional random field system with continuous $O(N)$ symmetry. The system undergos a paramagnetic-ferromagnetic transition above the lower critical dimension $d_{\rm lc} = 4$ for $N > N_c = 2.835$. Below the lower critical dimension and for $N < N_c$ the system exhibits a quasi-long-range order with zero order parameter and a power-law decay of correlations. Using functional renormalization group we obtain the surface scaling laws describing the ordinary surface transition for $d > d_{\rm lc}$ and the behavior of correlations near the surface in the quasi-long-range ordered phase for $d < d_{\rm lc}$

1. Introduction

Understanding of the critical properties of disordered systems attracted growing interest for decades. Among the most challenging problems is the critical behavior of the so-called random field systems in which the order parameter is linearly coupled to a random symmetry breaking field [19]. The effect of the random field (RF) disorder being more profound than many other types of disorder is much less understood. The prominent example is the random field Ising model (RFIM) whose complete understanding is still lacking despite significant numerical, analytical and experimental efforts [26]. The considerable progress has been achieved in recent years for the $O(N)$ symmetric random field models. These models are relevant for diverse physical applications including amorphous magnets [18], liquid crystals in porous media [3, 12], nematic elastomers [17], critical fluids in aerogels [7, 9, 24], vortices in type II superconductors [2], and stochastic inflation in cosmology [20]. It was found that the expansion around the lower critical dimension of the the RF $O(N)$ model $d_{\rm lc} = 4$ generates an infinite number of relevant operators whose flow can be studied using functional renormalization group [13, 15, 22, 31, 32]. Another challenging issue is the phase diagram of the RF systems below $d_{\rm lc}$. It is known that for the RF model true long-range order is forbidden below $d_{\rm lc} = 4$ [6]. Nevertheless, quasi-long-range order (QLRO) with zero order parameter and an

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infinite correlation length can persist even for $d_{\rm lc}^*(N) < d < d_{\rm lc}$, where $d_{\rm lc}^*(N)$ is the lower critical dimension for the paramagnetic-QLRO transition.

In general, the presence of boundaries in real systems can modify the behavior in the boundary region extended in the bulk only over distances of the order of the bulk correlation length. However, at the bulk critical point or in the QLRO phase, the bulk correlation length is infinite so that one can expect that the effect of boundaries is to be more pronounced. Indeed, the presence of the boundaries introduces a whole set of critical exponents describing the scaling behavior at and close to the boundary at criticality [1]. Several different classes of the surface transitions are known depending upon boundary conditions [23]. The different types of surface transitions have been studied for various systems with discrete and continuous symmetries using different methods, such as RG and numerical simulations [4, 28, 33]. However, not so much is known about the surface criticality in systems with RF disorder. The phase diagram of the 3D semi-infinite RFIM as a function of the ratio of bulk and surface interactions and the ratio of bulk and surface fields has been studied using a mean field approximation in [29]. The surface criticality of the RFIM has been studied numerically in [21]. It was also shown that the RF disorder on the surface of a 3D spin system with continuous symmetry destroys the long-range order in the bulk, and, instead, a QLRO emerges [14]. In this work we address the question of how the RF disorder in the bulk affect the behavior of spin systems with continuous symmetry in vicinity of free surfaces. We will consider the ordinary surface transition of the RF systems for $d > 4$ and the order parameter correlations in the QLRO phase near a free surface for $d < 4$.

2. Model

Let us consider a d-dimensional semi-infinite $O(N)$ spin system whose configuration is given by the N-component classical vector field $\mathbf{s}(\mathbf{r})$ satisfying the fixed-length constraint $|\mathbf{s}(\mathbf{r})|^2 = 1$. The position vector $\mathbf{r} = (\mathbf{x}, z)$ has a $(d-1)$-dimensional component $\mathbf{x}$ parallel to the surface and a one-dimensional component $z \geq 0$ that is perpendicular to the surface $z = 0$. It is convenient to introduce short notations for the volume integral over half space $\int_V := \int_0^\infty dz \int d^{d-1}x$ and for the surface integral $\int_S := \int d^{d-1}x$. The large-scale behavior of the disordered spin system can be described by the effective Hamiltonian

$$\mathcal{H}\left[\mathbf{s}\right] = \mathcal{H}_0\left[\mathbf{s}\right] + \mathcal{H}_{\rm surf}\left[\mathbf{s}\right] + \mathcal{H}_{\rm dis}\left[\mathbf{s}\right], \tag{1}$$

consisting of the sum of three terms which result from the semi-infinite bulk, surface and disorder in the bulk. The contributions from the semi-infinite bulk and the surface can be written in its simplest form as [5]:

$$\mathcal{H}_0\left[\mathbf{s}\right] = \int_V \left[\frac{1}{2}\left(\nabla \mathbf{s}(\mathbf{r})\right)^2 - \mathbf{h}\cdot\mathbf{s}(\mathbf{r})\right], \quad \mathcal{H}_{\rm surf}\left[\mathbf{s}\right] = -\int_S \mathbf{h}_1\cdot\mathbf{s}(\mathbf{x}), \tag{2}$$

where for simplicity we assume that the surface magnetic field $\mathbf{h}_1$ has the same direction as the bulk field $\mathbf{h}$. We consider a quite general type of bulk disorder such that its potential can be expanded in spin variables as follows

$$\mathcal{H}_{\rm dis}\left[\mathbf{s}\right] = -\int_V \sum_{\mu=1}^{\infty} \sum_{i_1...i_\mu} h^{(\mu)}_{i_1...i_\mu}(\mathbf{r}) s_{i_1}(\mathbf{r})...s_{i_\mu}(\mathbf{r}). \tag{3}$$

The coefficients $h^{(\mu)}_{i_1...i_\mu}(\mathbf{r})$ are Gaussian random variables with zero mean and variances given by

$$\overline{h^{(\mu)}_{i_1...i_\mu}(\mathbf{r}) h^{(\nu)}_{i_1...j_\nu}(\mathbf{r}')} \;=\; \delta^{\mu\nu}\delta_{i_1 j_1}...\delta_{i_\mu j_\nu} r_\mu \delta(\mathbf{r}-\mathbf{r}'). \tag{4}$$

The first two coefficients have simple physical interpretation: $h^{(1)}_i$ is a random magnetic field and $h^{(2)}_{ij}$ is a second-rank random anisotropy. The higher order coefficients $h^{(\mu)}$ are higher order random anisotropies. As was shown in [15], even if the system has only finite number of nonzero bare $h^{(\mu)}$, the RG transformations will generate an infinite set of higher-order anisotropies.

To average over disorder we use the replica trick and introduce n replicas of the original system. Averaging their joint partition function over disorder we obtain the replicated Hamiltonian as

$$\begin{aligned}\mathcal{H}_n \;=\; & \int_V \left\{ \sum_{a=1}^{n} \left[\frac{1}{2}\left(\nabla \mathbf{s}_a(\mathbf{r})\right)^2 - \mathbf{h}\cdot\mathbf{s}_a(\mathbf{r}) \right] - \right. \\ & \left. -\frac{1}{2T}\sum_{a,b=1}^{n} \mathcal{R}\left(\mathbf{s}_a(r)\cdot\mathbf{s}_b(\mathbf{r})\right) \right\} - \sum_{a=1}^{n}\int_S \mathbf{h}_1\cdot\mathbf{s}_a(\mathbf{x}),\end{aligned} \tag{5}$$

where we have defined the function $\mathcal{R}(z) = \sum_\mu r_\mu z^\mu$. The properties of the original disordered system (1) can be extracted in the limit $n \to 0$.

Power counting shows that $d_{\rm lc} = 4$ is the lower critical dimension of the model (5). Above the lower critical dimension the RF systems undergo a paramagnetic - ferromagnetic transition. The scaling behavior at criticality is controlled by a zero temperature fixed point (FP) similar to the RFIM, reflecting the fact that disorder dominates over the thermal fluctuations. However, the temperature is dangerously irrelevant. For instance, this results in violation of the usual hyperscaling relation and the appearance of a new universal exponent θ that modifies the hyperscaling relation to [26]:

$$\nu(d-\theta) = 2-\alpha, \tag{6}$$

where ν and α are the correlation length and the specific heat exponents. One also expects a dramatic slowing down as the transition is approached with the characteristic relaxation time $\ln\tau \sim t_1^{-\nu\theta}$, where $t_1 = |T-T_c|/T_c$ is the reduced temperature [16]. The magnetization in the bulk and on the surface vanish at the transition according to

$$\sigma(t_1) \sim t_1^{\beta}, \qquad \sigma_1(t_1) \sim t_1^{\beta_1}, \tag{7}$$

where we have introduced the bulk and the surface magnetization exponents. At the critical point $t_1 = 0$ small magnetic field in the bulk $\mathbf{h}$ and on the surface $\mathbf{h}_1$ can induce the magnetization in the bulk and also on the surface according to

$$\sigma(h) \sim h^{1/\delta}, \qquad \sigma_1(h) \sim h^{1/\delta_1}, \qquad \sigma_1(h_1) \sim h_1^{1/\delta_{11}}, \tag{8}$$

where we define the exponents δ, δ_1 and δ_{11}. Below the lower critical dimension d_{lc} a QLRO phase with zero magnetization can emerge. At criticality or in the QLRO phase, the correlation functions of the order parameter exhibit scaling behavior. Due to dangerous irrelevance of the temperature the connected and disconnected correlation functions scale with different exponents. We define the connected and disconnected correlation functions of the two local operators A and B as

$$[A(\mathbf{r}) \cdot B(\mathbf{r}')]_{\mathrm{con}} := \overline{\langle A(\mathbf{r}) \cdot B(\mathbf{r}')\rangle - \langle A(\mathbf{r})\rangle \cdot \langle B(\mathbf{r}')\rangle},$$
$$[A(\mathbf{r}) \cdot B(\mathbf{r}')]_{\mathrm{dis}} := \overline{\langle A(\mathbf{r})\rangle \cdot \langle B(\mathbf{r}')\rangle} - \overline{\langle A(\mathbf{r})\rangle} \cdot \overline{\langle B(\mathbf{r}')\rangle}.$$

Here the angular brackets denote the thermal averaging and the overbar stands for the disorder averaging. For instance, the connected and disconnected correlation functions of spins in the bulk scale independently as

$$[\mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{r}')]_{\mathrm{con}} \sim \frac{1}{|\mathbf{r} - \mathbf{r}'|^{d-2+\eta}}, \quad [\mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{r}')]_{\mathrm{dis}} \sim \frac{1}{|\mathbf{r} - \mathbf{r}'|^{d-4+\bar{\eta}}}. \tag{9}$$

Following the general scaling picture of the surface critical phenomena we introduce the surface exponents $\eta_\perp$ and $\bar{\eta}_\perp$ which replace the bulk exponents η and $\bar{\eta}$ in equations (9) when one of the points $\mathbf{r}$ or $\mathbf{r}'$ belongs to the surface:

$$[\mathbf{s}(\mathbf{x}, z) \cdot \mathbf{s}(\mathbf{x}', 0)]_{\mathrm{con}} \sim \frac{1}{((\mathbf{x} - \mathbf{x}')^2 + z^2)^{(d-2+\eta_\perp)/2}}, \tag{10}$$

$$[\mathbf{s}(\mathbf{x}, z) \cdot \mathbf{s}(\mathbf{x}', 0)]_{\mathrm{dis}} \sim \frac{1}{((\mathbf{x} - \mathbf{x}')^2 + z^2)^{(d-4+\bar{\eta})/2}}. \tag{11}$$

We also define the surface exponents $\eta_\parallel$ and $\bar{\eta}_\parallel$ that describe the connected and disconnected correlation function when both points lie on the surface:

$$[\mathbf{s}(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}')]_{\mathrm{con}} \sim \frac{1}{|\mathbf{x} - \mathbf{x}'|^{d-2+\eta_\parallel}}, \quad [\mathbf{s}(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}')]_{\mathrm{dis}} \sim \frac{1}{|\mathbf{x} - \mathbf{x}'|^{d-4+\bar{\eta}_\parallel}}. \tag{12}$$

3. Functional renormalization group

In the limit of low temperature and weak disorder the configuration of the system is fluctuating around the completely ordered state in which all replicas of all spins align along the same direction which is parallel to $\mathbf{h}$ and $\mathbf{h}_1$. It is convenient to split the order parameter $\mathbf{s}_a = (\sigma_a, \boldsymbol{\pi}_a)$ into the $(N-1)$-component vector $\boldsymbol{\pi}_a$ which is perpendicular to this direction and the component $\sigma_a = \sqrt{1 - \boldsymbol{\pi}_a^2}$ being parallel to this direction. Then the effective action of the system can be written as

$$\begin{aligned}\mathcal{S}[\boldsymbol{\pi}] &= \frac{1}{T}\sum_{a=1}^{n}\left\{\int_V \left[\frac{1}{2}(\nabla\boldsymbol{\pi}_a)^2 + \frac{(\boldsymbol{\pi}_a\cdot\nabla\boldsymbol{\pi}_a)^2}{2(1-\boldsymbol{\pi}_a^2)} - h\,\sigma_a\right] - \int_S h_1\,\sigma_a\right\} - \\ &\quad -\frac{1}{2T^2}\sum_{a,b=1}^{n}\int_V \mathcal{R}\left(\boldsymbol{\pi}_a\cdot\boldsymbol{\pi}_b + \sigma_a\sigma_b\right). \end{aligned} \tag{13}$$

In general one has to add to the action (13) the terms like $\delta^d(0)\int_V \ln(1-\boldsymbol{\pi}_a^2)$ generated by the Jacobian of the transformation from $\mathbf{s}_a$ to $\boldsymbol{\pi}_a$. However, in what follows we will use the dimensional regularization scheme [5] in which $\delta^d(0)=0$ so we can ignore these terms in action (13) from the beginning.

Let us denote averaging with the action (13) by double angular brackets and introduce the following correlation functions

$$G^{(L,K)}_{\alpha,\beta}(\mathbf{r},\mathbf{x}) = \left\langle\!\!\left\langle \prod_{\nu=1}^{L}\pi_{\alpha_\nu}(\mathbf{r}_\nu)\prod_{\mu=1}^{K}\pi_{\beta_\mu}(\mathbf{x}_\mu)\right\rangle\!\!\right\rangle, \tag{14}$$

where L points $\mathbf{r}=(\mathbf{r}_1,...,\mathbf{r}_L)$ are off surface and K points $\mathbf{x}=(\mathbf{x}_1,...,\mathbf{x}_K)$ are siting on the surface. In equation (14) we have used a short notation $\alpha=(\alpha_1,...,\alpha_L)$ where each α_ν stands for the component number i_ν and the replica number a_ν. The similar holds for β. Using correlation functions (14) one can compute the connected and disconnected functions defined in equations (9). However, since we are interested only in the scaling behavior it is more convenient to consider the similar correlation functions not for $\mathbf{s}$ but for $\boldsymbol{\pi}$ fields.

Expanding the effective action (13) in small $\boldsymbol{\pi}$ we will treat the quadratic part as a free action and the rest of the infinite series as interaction vertices. Then the correlation functions (14) can be expressed in terms of Feynman diagrams which give the low temperature and small disorder expansion. In practical calculations it is convenient to perform the Fourier transform with respect to $\mathbf{x}$: $\hat{\boldsymbol{\pi}}(\mathbf{q},z) = \int d^{d-1}x\boldsymbol{\pi}(\mathbf{x},z)e^{-i\mathbf{q}\cdot\mathbf{x}}$ and define $\int_q := \int d^{d-1}q/(2\pi)^{d-1}$. The quadratic terms give the free propagator

$$\hat{G}^{(0)}_q(z,z') = \frac{1}{2\bar{q}}\left[e^{-\bar{q}|z-z'|} + \frac{\bar{q}-h_1}{\bar{q}+h_1}e^{-\bar{q}(z+z')}\right], \tag{15}$$

where we have introduced the shorthand notation $\bar{q} := (q^2+h)^{1/2}$. The free surface corresponds to the limit $h_1\to 0$ in which equation (15) becomes the Neumann propagator consisting of the bulk part and the image part. In what follows we will use the Neumann propagator as the bare one and treat the terms proportional to h_1 as soft insertions [5, 11].

The correlation functions (14) calculated perturbatively in small disorder and temperature suffer from the ultraviolet divergences. To avoid mixture with infrared singularities in the $O(N)$-noninvariant correlation functions it is convenient to keep $\mathbf{h}\neq 0$. The ultraviolet divergences can be converted into poles in $\varepsilon = d-4$ using dimensional regularization. To renormalize the theory one has to absorb these poles into finite number of Z-factors. However, all the Taylor coefficients r_μ of the disorder correlator $\mathcal{R}(\phi)$ turn out to be relevant operators so that one has to introduce renormalization of the whole function. Since the scaling behavior is controlled by a zero temperature FP we will disregard all terms involving more than two replicas which are suppressed in the limit $T \to 0$. The renormalization of the disorder simplifies by changing variables: $\mathcal{R}(\phi) = R(z)$ where $z = \cos\phi$, for instance, $\mathcal{R}'(1) = -R''(0)$. In terms of the variable ϕ, the function $R(\phi)$ becomes periodic with period 2π in the RF case. The relation between the renormalized and the bare correlation functions reads

$$G^{(L,K)}(\mathbf{r}; T, h, h_1, R, \mu) \;=\; Z_\pi^{-(L+K)/2} Z_1^{-K/2} \mathring{G}^{(L,K)}(\mathbf{r}; \mathring{T}, \mathring{h}, \mathring{h}_1, \mathring{R}). \tag{16}$$

where circles denote the bare quantities and μ is an arbitrary momentum scale. The ultraviolet divergences are absorbed into Z-factors according to

$$\mathring{\boldsymbol{\pi}} = Z_\pi^{1/2}\boldsymbol{\pi}, \qquad \mathring{\boldsymbol{\pi}}\big|_s = (Z_\pi Z_1)^{1/2}\, \boldsymbol{\pi}\big|_s, \qquad \mathring{h} = \mu^2 Z_T Z_\pi^{-1/2} h, \tag{17}$$

$$\mathring{h}_1 = \mu Z_T (Z_\pi Z_1)^{-1/2} h_1, \quad \mathring{T} = \mu^{2-d} Z_T T, \qquad \mathring{R} = \mu^{4-d} K_d^{-1} Z_R[R], \tag{18}$$

where $(2\pi)^d K_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a d-dimensional unit sphere and $\Gamma(x)$ is the Euler gamma function. In equation (18) $Z_R[R]$ is a functional acting on the renormalized disorder correlator $R(\phi)$ which has the following loop expansion:

$$Z_R[R] = R + \delta^{(1)}(R,R) + \delta^{(2)}(R,R,R) + ..., \tag{19}$$

where $\delta^{(1)}(R,R)$ is bilinear in R and proportional to $1/\varepsilon$, while $\delta^{(2)}(R,R,R)$ is cubic in R and contains terms of order $1/\varepsilon$ and $1/\varepsilon^2$. According to equations (17) the surface field $\boldsymbol{\pi}|_s$ renormalizes differently from the field $\boldsymbol{\pi}$ in the bulk. The new factor Z_1 serves to cancel the additional ultraviolet divergences in Feynman diagrams arising from the image part of the Neumann propagator $\hat{G}_q^{(0)}(z,z')$ for $z' \to 0$. The renormalized theory is not unique and depends on the scale μ. Using this fact we will derive the functional renormalization group equation.

We now consider how the scaling behavior can be extracted from the renormalized theory. Using the independence of the bare theory on the momentum scale μ one can derive the flow equations for the renormalized correlation functions differentiating both sides of equation (16) with respect to μ at fixed bare quantities. One finds that the renormalized correlation functions satisfy the following FRG equation

$$\Big[\mu\partial_\mu + (d-2-\zeta_T)T\partial_T - \zeta_h h\partial_h - \zeta_{h_1} h_1\partial_{h_1} + \frac{L}{2}\zeta_\pi + \\ + \frac{K}{2}(\zeta_\pi + \zeta_1) - \int d\phi\, \beta[R(\phi)] \frac{\delta}{\delta R(\phi)}\Big]\, G^{(L,K)} = 0, \tag{20}$$

where the integral in the last line is taken over a period, i.e., $(0,\pi)$ for RA and $(0,2\pi)$ for RF models and we have introduced the scaling functions:

$$\zeta_i = \mu\partial_\mu \ln Z_i\big|_0, \quad (i = T, \pi, 1), \tag{21}$$

$$\zeta_h = 2 + \zeta_T - \zeta_\pi/2, \quad \zeta_{h_1} = 1 + \zeta_T - (\zeta_\pi + \zeta_1)/2, \tag{22}$$

$$\beta[R] = -\,\mu\partial_\mu R(\phi)\big|_0. \tag{23}$$

Here the zero indicates that the derivatives are taken at fixed bare quantities. Flow equations similar to equation (20) hold also for the correlation functions in which some or all the fields $\boldsymbol{\pi}_a(\mathbf{r})$ are replaced by $\sigma_a(\mathbf{r})$ and for other observables, e.g., the correlation length and the magnetization.

The long-distance physics can be obtained from the solution of the FRG equation (20) in the limit of $\mu \to 0$. The renormalized disorder correlator and the temperature flow according to

$$-\mu\partial_\mu R(\phi) = \beta[R], \tag{24}$$

$$-\mu\partial_\mu \ln T = 2 - d + \zeta_T. \tag{25}$$

The scaling behavior is controlled by a zero temperature FP $\beta[R^*] = 0$ with R^* of order ε and $T^* = 0$. Indeed, according to equation (25), the temperature is irrelevant, i.e. it flows to 0 in the limit $\mu \to 0$ for $d > 2$ and for sufficiently small $\zeta_T = O(R)$. Although one expects that ζ_T is small in the vicinity of the FP, one has to take caution whether the zero temperature FP survives in three dimensions where $\zeta_T \sim \varepsilon$ is negative [13]. The stability of the FP can be checked by computing the eigenvalues of the disorder flow equation (24) linearized about the FP solution: $R(\phi) = R^*(\phi) + \sum_i t_i \Psi_i(\phi)$. Since one expects that for $d > 4$ ($\varepsilon > 0$) the FP $R^*(\phi)$ describes the paramagnetic-ferromagnetic transition it has to be unstable in a single direction $\Psi_1(\phi)$ with eigenvalue $\lambda_1 > 0$: $\beta[R^* + t_1\Psi_1] = \lambda_1 t_1 \Psi_1 + O(t_1^2)$. In the vicinity of the zero temperature FP that controls the paramagnetic-ferromagnetic transition, the FRG equation for the correlation length ξ can be written as

$$\left[\mu\partial_\mu - \lambda_1 t_1 \frac{\partial}{\partial t_1}\right] \xi(\mu, t_1) = 0. \tag{26}$$

Dimensional analysis implies that $\xi(\mu, t_1) = \mu^{-1}\bar{\xi}(t_1)$. This reduces equation (26) to an ordinary differential equation (ODE) whose solution is $\xi \sim \mu^{-1} t_1^{-1/\lambda_1}$. The latter describes divergence of the correlation length on the critical line at zero temperature when the strength of disorder approaches the critical value. Assuming that along the transition line at finite temperature $t_1 \sim T - T_c$ we find that the positive eigenvalue λ_1 gives the critical exponent of the correlation length $\nu = 1/\lambda_1$. For $d < 4$ ($\varepsilon < 0$) the FP becomes stable and describes a QLRO phase. The fluctuations exhibit power-law correlations in the whole QLRO phase so that the correlation length ξ is always infinite down to the lower critical dimension of the QLRO - paramagnetic transition.

Let us consider the solution of equation (20) for the connected two-point correlation functions. The dangerous irrelevance of the temperature manifests itself in the fact that the connected (bulk or surface) two point functions are proportional to T in the low temperature limit. Thus, setting $h = h_1 = 0$ and $R = R^*$ we can rewrite equation (20) as

$$\left[\mu\partial_\mu + \frac{1}{2}(L+K)\zeta_\pi^* + \frac{K}{2}\zeta_1^* + \theta\right] G_{\text{con}}^{(L,K)} = 0, \tag{27}$$

where the asterisk denotes that the function is computed at the FP. In equation (27) we have defined the exponent

$$\theta = d - 2 - \zeta_T^*, \tag{28}$$

which describes the flow of the temperature (25) in the vicinity of the FP and has been introduced ad hoc in the modified hyperscaling relation (6). Using the method of characteristics and dimensional analysis one can write the solution of equation (27) in the form

$$G_{\text{con}}^{(L,K)}(rb; R^*) = b^{-(\frac{1}{2}(L+K)\zeta_\pi^* + K\zeta_1^*/2 + \theta)} f_c(r; R^*). \tag{29}$$

Considering the connected two point functions (29) with $(L = 2, K = 0)$, $(L = 1, K = 1)$, and $(L = 0, K = 2)$ we derive the critical exponents:

$$\eta = \zeta_\pi^* - \zeta_T^*, , \quad \eta_\perp = \zeta_\pi^* + \zeta_1^*/2 - \zeta_T^*, \quad \eta_\parallel = \zeta_\pi^* + \zeta_1^* - \zeta_T^*. \tag{30}$$

We next turn to the disconnected two-point correlation functions. At variance with the connected correlation functions they are not proportional to the temperature. Thus, at $h = h_1 = T = 0$ they satisfy the same equation (27) but without the term θ in large square brackets. The solution of the latter FRG equation is given by

$$G_{\text{dis}}^{(L,K)}(rb; R^*) = b^{-(\frac{1}{2}(L+K)\zeta_\pi^* + K\zeta_1^*/2)} f_d(r; R^*). \tag{31}$$

Repeating the analysis we did for the connected functions, we arrive at

$$\bar\eta = 4 - d + \zeta_\pi^* = 2 + \eta - \theta, \quad \bar\eta_\perp = 4 - d + \zeta_\pi^* + \zeta_1^*/2 = 2 + \eta_\perp - \theta, \tag{32}$$

$$\bar\eta_\parallel = 4 - d + \zeta_\pi^* + \zeta_1^* = 2 + \eta_\parallel - \theta. \tag{33}$$

Note that the exponents (30) and (32)–(33) are related by

$$2\eta_\perp = \eta + \eta_\parallel, \qquad 2\bar\eta_\perp = \bar\eta + \bar\eta_\parallel. \tag{34}$$

Finally we study the profile of the spontaneous magnetization below and at the paramagnetic-ferromagnetic transition for $d > d_{\text{lc}}$. The magnetization as a function of the distance to the surface z, the reduced temperature t_1, and the bulk and surface magnetic fields h and h_1 satisfies the following flow equation

$$\left[\mu\partial_\mu - \zeta_h^* h\partial_h - \zeta_{h_1}^* h_1\partial_{h_1} + \frac{1}{2}\zeta_\pi^* + \frac{j}{2}\zeta_1^* - \lambda_1 t_1 \frac{\partial}{\partial t_1}\right]\sigma(z, t_1, h, h_1) = 0. \tag{35}$$

Here $j = 0$ and $z > 0$ corresponds to the bulk magnetization σ while $j = 1$ and $z = 0$ gives the surface magnetization σ_1. The solution of equation (35) can be written as

$$\sigma(z, t_1, h, h_1) \;=\; b^{-(\frac{1}{2}\zeta_\pi^* + \frac{j}{2}\zeta_1^*)}\,\sigma(zb^{-1}, t_1 b^{\lambda_1}, hb^{\zeta_h^*}, h_1 b^{\zeta_{h_1}^*}). \tag{36}$$

We first consider the profile for $h = h_1 = 0$. The solution (36) interpolates between the surface magnetization $\sigma_1(t_1) \sim t_1^{(\zeta_\pi^*+\zeta_1^*)/(2\lambda_1)}$ at $z \approx 0$ and the bulk magnetization $\sigma(t_1, z) \sim t_1^{\zeta_\pi^*/(2\lambda_1)}$ for $z \gg \xi$. Reexpressing the latter in terms of ν,$\bar{\eta}$, and $\bar{\eta}_\parallel$ we obtain that the bulk and the surface magnetization exponents defined in equation (7) are given by

$$\beta = \frac{1}{2}\nu(d-4+\bar{\eta}), \quad \beta_1 = \frac{1}{2}\nu(d-4+\bar{\eta}_\parallel). \tag{37}$$

At the critical point $t_1 = 0$ and finite external fields we find that $\sigma(h) \sim h^{\zeta_\pi^*/(2\zeta_h^*)}$ in the bulk and $\sigma_1(h) \sim h^{(\zeta_\pi^*+\zeta_1^*)/(2\zeta_h^*)}$ or $\sigma_1(h_1) \sim h_1^{(\zeta_\pi^*+\zeta_1^*)/(2\zeta_{h_1}^*)}$ at the surface. Thus, the exponents δ, δ_1, and δ_{11} defined in equations (8) satisfy the following scaling relations:

$$\frac{\delta-1}{2-\eta} = \frac{\nu}{\beta}, \quad \frac{\delta_1-\beta/\beta_1}{2-\eta} = \frac{\nu}{\beta_1}, \quad \frac{\delta_{11}-1}{1-\eta_\parallel} = \frac{\nu}{\beta_1}. \tag{38}$$

4. Surface exponents to one-loop order

We now renormalize the both semi-infinite RF and RA models to one-loop order and explicitly calculate the surface critical exponents to first order in $\varepsilon = d-4$. The factors Z_π, Z_T and $Z_R[R]$ defined in equations (17)–(19) are the same that appear in the case of the infinite systems. They have been calculated in several works up to two-loop order [13, 15, 22, 31]. To one-loop order they read

$$Z_\pi = 1-(N-1)\frac{R''(0)}{\varepsilon}+O(R^2), \quad Z_T = 1-(N-2)\frac{R''(0)}{\varepsilon}+O(R^2), \tag{39}$$

$$\varepsilon\delta^{(1)}(R,R) = \frac{1}{2}R''(\phi)^2 - R''(0)R''(\phi) - (N-2)\Big\{R''(0)[2R(\phi) + \\ +R'(\phi)\cot\phi] - \frac{1}{2\sin^2\phi}\big[R'(\phi)\big]^2\Big\}. \tag{40}$$

The new factor Z_1 that eliminates the poles resulting from the presence of the surface can be determined from the renormalization of the two point function $\mathring{G}^{(1,1)}(p, z; \mathring{h}, \mathring{T}, \mathring{R})$ which reads to one-loop order

$$\mathring{G}^{(1,1)}(p,z;\mathring{h},\mathring{T},\mathring{R}) = \mathring{T}\frac{e^{-\bar{p}z}}{\bar{p}}\left\{1-\frac{K_d}{4\varepsilon}\mathring{R}''(0)\times \right. \\ \left. \times\left[(N-3)\left(\frac{\mathring{h}}{\bar{p}^2}+\frac{z\mathring{h}}{\bar{p}}\right)+2(N+1)\right]+O(\mathring{R}^2)\right\}, \tag{41}$$

where $\bar{p} = (p^2+\mathring{h}^2)^{1/2}$. The factor Z_1 can be found from the renormalization condition

$$Z_\pi^{-1}Z_1^{-1/2}\mathring{G}^{(1,1)}(p,z;\mathring{h},\mathring{T},\mathring{R}) = \text{finite for } \varepsilon\to 0, \tag{42}$$

where the bare $\mathring{h}, \mathring{T}, \mathring{R}$ are replaced by the renormalized h, T and R according to equations (17)-(18). We obtain

$$Z_1 = 1 - (N-1)\frac{R''(0)}{\varepsilon} + O(R^2). \tag{43}$$

Thus, to one loop order we have $Z_1 = Z_\pi + O(R^2)$. Using equations (21)-(23) we calculate the scaling functions

$$\zeta_T \;=\; -(N-2)R''(0) + O(R^2), \;\; \zeta_\pi = \zeta_1 = -(N-1)R''(0) + O(R^2), \tag{44}$$

and the beta function

$$\begin{aligned}\beta[R] \;=\; & -\varepsilon R(\phi) + \frac{1}{2}R''(\phi)^2 - R''(0)R''(\phi) - (N-2)\times \\ & \times\left\{R''(0)[2R(\phi) + R'(\phi)\cot\phi] - \frac{1}{2\sin^2\phi}\big[R'(\phi)\big]^2\right\} + O(R^2)\end{aligned} \tag{45}$$

to one-loop order. The solution of the FP equation $\beta[R^*] = 0$ with the beta function (45) has been analyzed for different values of N and different sign of ε in [13, 22, 31]. We first assume for granted that the flow has a FP $R^*(\phi)$ which is a π-periodic function for the RA model and a 2π-periodic function for the RF model. Then, the surface critical exponents can be computed to one loop using equations (30) and (32)-(33). They give

$$\eta = -R^{*\prime\prime}(0), \qquad \bar\eta = -\varepsilon - (N-1)R^{*\prime\prime}(0), \tag{46}$$

$$\eta_\perp = -\frac{N+1}{2}R^{*\prime\prime}(0), \quad \bar\eta_\perp = -\varepsilon - \frac{3}{2}(N-1)R^{*\prime\prime}(0), \tag{47}$$

$$\eta_\parallel = -NR^{*\prime\prime}(0), \qquad \bar\eta_\parallel = -\varepsilon - 2(N-1)R^{*\prime\prime}(0). \tag{48}$$

The other surface exponents are related to the exponents (46)-(48) by the scaling relations (37) and (38).

4.1. Paramagnetic-ferromagnetic transition for $d > 4$ ($\varepsilon > 0$)

The RF model is described by $R(\phi)$ which is a 2π-periodic function. Numerical solution of the FP equation shows that for $d > 4$ a 2π-periodic solution exists only for $N > N_c = 2.834\,74$. It has $R^{*\prime\prime}(0) < 0$ and it disappears when $N \to N_c^+$. This cuspy FP is once unstable with the positive eigenvalue $\lambda_1 = \varepsilon$. Thus, the correlation length exponent $\nu = 1/\varepsilon + (\varepsilon^0)$ coincides with the DR prediction to one-loop order. Remarkably, the non-zero $R^{*\prime\prime\prime}(0^+)$ vanishes for $N > N^* = 18 + O(\varepsilon)$. The non-analyticity becomes weaker as N increases and starts with $R^{*(2p(N)+1)}(0^+) \neq 0$ where $p \sim N$ [22, 30, 31]. Weaker non-analyticity results in restoring the DR critical exponents for $N > N^*$. The critical exponents η_i and $\bar\eta_i$ computed using equations (46)-(48) as functions of N are shown in the right panel of Figure 1. With increasing N they monotonically decay approaching the DR values at $N = N^*$ and satisfying the inequalities: $\eta < \bar\eta < \eta_\perp < \bar\eta_\perp < \eta_\parallel < \bar\eta_\parallel$. The bulk and surface magnetization exponents β and β_1 calculated for different N

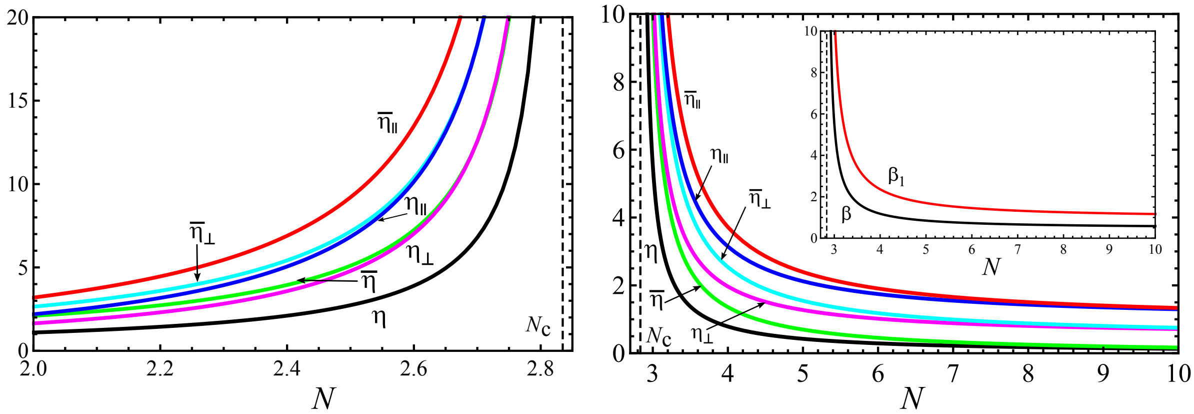

Figure 1. **Left panel:** the critical exponents η_i and $\bar{\eta}_i$ (divided by $|\varepsilon|$), which describe the power-law decay of correlations in the QLRO phase of the RF model below the lower critical dimension, as functions of N for $N < N_c$. **Right panel:** the critical exponents η_i and $\bar{\eta}_i$ (divided by ε), which describe the paramagnetic-ferromagnetic transition of the RF model above the lower critical dimension, as functions of N for $N > N_c$. **Inset:** the bulk magnetization exponent β and the surface magnetization exponent β_1 as functions of N.

are shown in the inset of the right panel of Figure 1. To one-loop order they obey the relation $\beta_1 = 2\beta$. Up to now both magnetization exponents have been studied only for the 3D RFIM where numerical simulations give $\beta = 0.0017 \pm 0.005$ [25] and $\beta_1 = 0.23 \pm 0.03$ [21]. Thus, the ratio β_1/β for the RF $O(N)$ systems in $d > 4$ is much smaller than for the 3D RFIM.

4.2. Quasi-long-range order for $d < 4$ ($\varepsilon < 0$)

Below the lower critical dimension the flow equation for the disorder correlator has an attractive 2π-periodic FP solution. This cuspy FP appears only for $2 \leq N < N_c$ where it controls the scaling behavior of spin fluctuations in the QLRO phase. The corresponding exponents η_i and $\bar{\eta}_i$ as functions of N are shown in the left panel of Figure 1. In the case $N = 2$ the FP equation admits for an explicit non-analytic ϕ_0-periodic solution given by

$$R^*(\phi) = \frac{|\varepsilon|\phi_0^4}{72}\left[\frac{1}{36} - \left(\frac{\phi}{\phi_0}\right)^2\left(1 - \frac{\phi}{\phi_0}\right)^2\right]. \tag{49}$$

Using equations (46)-(48) one obtains

$$\eta = \frac{\phi_0^2}{36}|\varepsilon|, \qquad \bar{\eta} = \left(1 + \frac{\phi_0^2}{36}\right)|\varepsilon|, \qquad \bar{\eta}_\perp = \left(1 + \frac{\phi_0^2}{24}\right)|\varepsilon|, \tag{50}$$

$$\eta_\parallel = \frac{\phi_0^2}{18}|\varepsilon|, \qquad \bar{\eta}_\parallel = \left(1 + \frac{\phi_0^2}{18}\right)|\varepsilon|, \quad \eta_\perp = \frac{\phi_0^2}{24}|\varepsilon| \tag{51}$$

with $\phi_0 = 2\pi$ for the RF system. The semi-infinite RF $O(2)$ model can be mapped onto a semi-infinite periodic disordered elastic system with a free surface. There is one to one correspondence between the Bragg glass phase of the elastic system and the QLRO phase of the studied spin model. The power-law decay of the spin correlations in the QLRO phase corresponds to the logarithmic growth of the displacements in the disordered elastic system. Moreover, the exponents η, $\eta_\perp$ and $\eta_\parallel$ provide the universal amplitudes of the logarithmic growth of the displacements in the bulk, at the surface and along the surface, respectively. For a ϕ_0-periodic elastic system with a free surface these amplitudes are given by equations (50)-(51). In particular, we find that the logarithmic growth of the displacements along the surface is twice large as the logarithmic growth in the bulk. In the case when only one point is on the surface the growth is enhanced by 50%. The presence of a free surface can be considered as an extended defect of a special kind. The influence of potential-like extended defects on the Bragg-glass has been recently studied in [10, 27].

5. Summary

In the present work we have studied the RF semi-infinite $O(N)$ systems with a free surface. Above the lower critical dimension $d_{\mathrm{lc}} = 4$ the systems undergo a paramagnetic-ferromagnetic transition for $N > N_c$, while below d_{lc} and for $N < N_c = 2.835$ they exhibit a QLRO phase with zero magnetization and power-law correlation of fluctuations. Using FRG we have derived the surface scaling behavior at criticality as well as in the QLRO phase, and calculated the corresponding surface exponents to lowest order in $\varepsilon = d - 4$. We have found that the dimensional reduction prediction for the surface scaling is broken similar to what happens in the bulk. We have shown that the connected and disconnected correlation functions scale differently also at the surface and derived the scaling relations between different surface exponents. The surface exponents obtained for the 3D RF $O(2)$ can be used to describe the growth of displacements near a free surface in semi-infinite periodic elastic systems in disordered media. The methods developed in this work can be also applied to the systems with random anisotropy disorder [8].

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