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BOUNDED SHORTENING IN COXETER COMPLEXES AND BUILDINGS

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We prove that the standard generating set for a Coxeter group of finite rank satisfies "falsification by a fellow traveller property" in the sense of W. Neumann and M. Shapiro. In particular this implies that the geodesic words in standard generators form a regular language. Similar property established for buildings and this implies rationality of growth series for certain groups acting on buildings.

Introduction

W. Neumann and M. Shapiro have shown that any geometrically finite hyperbolic group G has a generating set A so that the geodesic words in generators A form a regular language and the growth function is rational (Theorem 4.3 in [7]). This is done by using a criterion which essentially goes back to [4], namely, that any word in generators A which is not geodesic has a close neighbour which is shorter. In [7] this criterion is called "falsification by a fellow traveller" (FFT–property). Clearly this property can be defined for any graph with a graph metric. We prove that FFT property holds in the Cayley graph of a Coxeter group with respect to a standard generating set. We prove that the dual graph of any building satisfies FFT property. This implies that that the growth function of the groups acting simply transitively on the chambers of building is rational with respect to some generating set.

1. FFT–property

Definitions. Let Γ be a connected graph. A path joining the vertex x to the vertex y is a map $p : \{0, 1, 2, \ldots, n_p\} \to$ Vert $\Gamma, n_p \in \mathbb{N}$, where $p(0) = x, p(y) = y$ and all subsequent pairs of vertices are incident. For convenience, we often consider p as an ultimately constant map from $\mathbb{N}$ to Vert Γ, making p stopped after the moment n_p. We say that the edge paths with the same extremities δ-fellow travel for $\delta \in \mathbb{N}$ if the distance $d(w(t), v(t))$ never exceeds δ. We say that Γ has the falsification by fellow traveller property (or FFT– property) if there is a δ such that for any non-geodesic edge path in Γ there exists a shorter path with the same value that δ-fellow travels

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it. Let G be a finitely generated group and A a finite set and $a \mapsto \bar{a}$ a map of A to a monoid generating set $\bar{A} \subset G$. As is usual, A^* denotes the free monoid on A and the natural projection $A^* \to G$ is denoted $w \mapsto \bar{w}$. Any subset L of A^* which surjects onto G is called a normal form for G. The Cayley graph $\mathcal{C}_A(G)$ is the directed graph with vertex set G and a directed edge from g to $g\bar{a}$ for each $g \in G$ and $a \in A$; we give this edge a label a. We require that $\bar{A} = \bar{A}^{-1}$.

The following proposition explains our interest to the FFT–property ([7], Prop. 4.2).

Proposition. *If A has the falsification by fellow traveller property then the growth function of G with respect to A is rational.*

2. Walls in Coxeter complexes

We recall some basic definitions about Coxeter systems. For more about them see [Hi] or [Bo].

Definitions. A pair (W, S) is called a Coxeter system (of finite type) if W is a group with a finite subset S such that W has the presentation

$$\langle s : s \in S | (ss')^{m_{ss'}} = 1 \text{ when } m_{ss'} < \infty \rangle$$

where $m_{ss'} \in \{1, 2, 3, \ldots, \infty\}$ is the order of ss', and $m_{ss'} = 1$ if and only if $s = s'$. Let $\mathcal{C} = \mathcal{C}(\mathcal{W})$ be a Cayley graph of a Coxeter group W with respect to the standard generating system S. We call the edge $w \xrightarrow{s} ws$ to be inverse to the edge $ws \xrightarrow{s} w$ and we will call the pair of mutually inverted edges by a combinatorial edge and denote it by $\{w, ws\}$. The group W acts on the left on $\mathcal{C}$ by isometries. For any involution $w \in W$ we define its wall H_w as the set of all edges, inverted by w. The edge path $e_1 e_2 \cdots e_n$ is said to cross the wall H if at least one of its edges belongs to H. Each wall H separates $\mathcal{C}$ into two connected components H^+, H^- which are full subgraphs and each edge path connecting the vertices from different components crosses H. Two walls H_u, H_v are parallel if the element uv is of infinite order.

We can now state the Parallel Wall Theorem.

Theorem 1. *There exists a constant $K > 0$ such that for each point vertex x of $\mathcal{C}$ and for each wall H distance at least K from x, there exists another wall H' parallel to H which separates x from H. Moreover this wall can be chosen parallel to H.*

Clearly, the Parallel Wall Theorem implies the Separating Wall Theorem stated below.

Theorem 2. *There exists a constant $K > 0$ such that for each point $x \in \mathcal{C}$ and any wall H distance at least K from x, there exists another wall H' which separates x from H. Moreover this wall can be chosen parallel to H.*

In [1] the dominance relation is defined on the set of roots of (W, S). The dominance can be translated to geometry of walls as follows. Each wall H

divides the Cayley graph into two halfspaces H^+, H^- – we choose as the positive halfspace H^+ those one which does not contained 1. Then the dominance relation on the set of walls is just the containment relation on the set of positive halfspaces. It is shown in [2] that both assertions above and moreover they are equivalent to the following Finiteness Theorem which is proven in [1].

Theorem 3. *For any finitely generated Coxeter group the set of maximal positive halfspaces (relative to the containment) is finite.*

3. Falsification in Coxeter groups

We recall the construction of a Coxeter complex Σ for a Coxeter system (W, S) [3], [5]. By a special subgroup of W we mean a subgroup $\langle T\rangle$, generated by a proper subset $T \subset S$. The vertices of Σ are in one one correspondence with the left cosets of maximal special subgroups. More generally, the k-simplices of Σ are the left cosets of the special subgroups of rank $|S|-1-k$. In particular, the top-dimensional simplices (=chambers) are the cosets of the trivial subgroup of W, that is the elements of W. The codimension one simplexes (=panels) are the cosets of cyclic special subgroups. The incidence relation between the simplices is given by the containment relation between cosets. For example

$$v\langle s\rangle \text{ is the panel of } w \Longleftrightarrow w = vs \text{ or } w = v,$$

thus $v\langle s\rangle$ is the panel of exactly two chambers: v and vs. Define the dual graph of Σ with the set of all chambers W as vertices and panels as the edges – the ends of the edge are the chambers, adjacent along the corresponding panel. Thus the chambers v, w are adjacent iff they have the panel $u\langle s\rangle$ in common that is $w = vs$ or $v = ws$. We conclude that $\mathcal{C}$ is the modified Cayley graph of W with respect to generating system S [5]. The modification consists of the identification the edge $w \xrightarrow{s} ws$ with its inverse $ws \xrightarrow{s} w$. W acts simplicially on the Coxeter complex and this induces the standard action on the Cayley graph.

The edge paths in the Cayley graph are in the one one correspondence with the nonstuttering galleries in the Coxeter complex. The notion of the wall in the Cayley graph, being translated into the Coxeter complex, means the standard notion of the wall there – namely replacing the edges of Cayley wall by the corresponding panels we get the Coxeter wall.

Theorem 4. *The standard generating set S of any Coxeter group (W, S) satisfies the falsification by a fellow traveller property.*

Proof. We have to prove that any gallery $\Gamma = C_1C_2\cdots C_n$ in $\mathcal{C}$ which is not geodesic has a uniformly closed neighbour which is shorter. Take a subgallery $\Gamma' = C_i\cdots C_j$ which is not geodesic but any proper subgallery of which is already geodesic. It is well known fact that in a Coxeter complex the gallery is geodesic iff it crosses each wall at most twice, see e. g. [3]. Hence there is a wall H which is crossed by the subgallery Γ' at least twice. Indeed Γ' crosses H exactly twice since if some proper subgallery of Γ' would cross H twice then it would not be geodesic, contrary to the choice of Γ'. It follows that the chambers C_i and C_j lie on the same side of H, say H^-, and the subgallery Γ'' of Γ' obtained by deleting C_i and C_j lies on the another side, say H^+.

Γ' *lies in a* K*– neighbourhood of* H *where* K *is a constant in a Separating Wall Theorem.*

Suppose not, then there is a chamber $C \in \mathcal{C}$ at a distance greater than K from H. By the Parallel Wall Theorem there is the wall H_1 separating C from H. In particular, H_1 is contained in H^+. Let H_1^+ be those halfspace of H_1 which is contained in H^+. Then $C \subset H_1^+$ and $C_{i+1}, C_{j-1} \subset H_1^-$. Hence Γ'' have to cross H' at least twice and thus is not geodesic - contradiction.

Let $H = H_w$ for some reflection $w \in W$ and consider the gallery $w\Gamma''$. Clearly, it has the same origin and the end as Γ' does. But it is shorter than Γ' since it does not contain C_i, C_j. We assert that $w\Gamma''$ δ–fellow travels Γ' for $\delta = 2K + 2$. This immediately follows from the fact that $d(x, wx) \leq 2K$.

4. Falsification in buildings

Given a building Δ, there is a metric on the set of chambers of Δ, and we will want a path metric space which reflects this metric [6]. To do this we let Δ' be the graph dual to Δ. That is to say, the vertices of Δ' are the barycenters of the chambers of Δ. Two such vertices are connected by an edge when they lie in chambers with a common face. As usual, Δ' is metrized considering each edge as isometric to the unit interval. Non-stuttering galleries of Δ correspond to edge paths in Δ'. The decomposition of Δ into apartments induces a decomposition of Δ' into apartments which are isometric as labelled graphs to the Cayley graph of (W, S), the Coxeter system of Δ.

Theorem 5. *The dual graph of any locally finite building satisfies FFT property.*

Proof. Thus we have to prove that any nonstuttering gallery $\Gamma = C_1 C_2 \cdots C_n$ in $\mathcal{C}$ which is not geodesic has a uniformly closed neighbour which is shorter. Take a subgallery $\Gamma' = C_i C_{i+1} \cdots C_j$ which is not geodesic but any proper subgallery of which is already geodesic. In particular $\Gamma'' = C_i C_{i+1} \cdots C_{j-1}$ is geodesic. Let Σ be an apartment, containing both C_i and C_{j-1}. Any apartment Σ in any building Δ is convex in a sense that any geodesic gallery in Δ with both extremities in Σ is entirely contained in Σ [3], IY,4. In particular Γ'' is entirely contained in Σ. We let $\rho_{\Sigma,C}$ be the canonical retraction onto Σ centered at chamber C. (See, for example [Brown, IV.3].) It can be characterized as the unique chamber map $\Delta \to \Sigma$ which fixes C pointwise and maps every apartment containing C isomorphically onto Σ. We use the fact that ρ does not increase distance. Take $\rho = \rho_{\Sigma, C_{j-1}}$. Note that C_j is not folded by ρ onto C_{j-1} since these two chambers are contained in some apartment which mapped by ρ isomorphically onto Σ. Hence the length of the gallery $\rho(\Gamma) = C_i C_{i+1} \cdots C_{j-1} \rho(C_j)$ is the same as that of Γ and clearly Γ and $\rho(\Gamma)$ 1 -fellow travel each other. Thus it is enough to prove that $\rho(\Gamma)$ can be boundedly shortened. But this is proven above for the Coxeter complex.

Corollary. *Suppose Δ is a building whose apartments are isomorphic to the Coxeter complex Σ of a Coxeter system and that G is a finitely generated group which acts simplicially and simply transitively on the chambers of Σ. Let A be a generating set of G consisting of elements moving the fixed base vertex of X distance one apart relative to a graph metric on a 1-skeleton of Δ. Then A satisfies the falsification by fellow traveller property. In particular, the set of A–geodesic words forms a regular language and the growth function of G with respect to A is rational.*

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